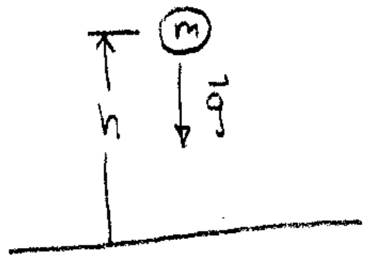
I. ELECTRIC POTENTIAL

(A) RECALL FROM PHYSICS 1A THE GRAVITATIONAL POTENTIAL ENERGY OF A MASS M IN A UNIFORM GRAVITATIONAL FIELD



$$PE = mgh$$

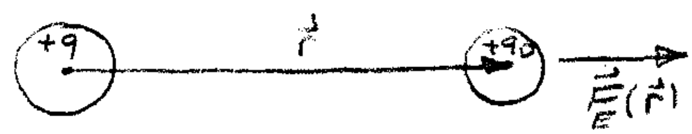
$$\text{Total Energy} = E = KE + PE = \text{const.}$$

$$\Delta E = 0$$

$$\Delta KE + \Delta PE = 0$$

$$\Delta KE = -\Delta PE$$

(B) ELECTRIC POTENTIAL ENERGY IS SIMILAR



q_0 experiences a force pushing it this way

$\Delta KE > 0 \Rightarrow \Delta PE < 0$

To put q_0 at $\vec{r}$, we had to bring it in from a distance by exerting a force

$$\vec{F}_A = -\vec{F}_E$$

THE WORK DONE ON THE SYSTEM IS STORED AS POTENTIAL ENERGY

$$dW = \vec{F}_A \cdot d\vec{r}$$

$$dW = -\vec{F}_E \cdot d\vec{r}$$

But $\quad \vec{F}_E = q_0 \vec{E}$

$$dW = -q_0 \vec{E} \cdot d\vec{r}$$

The TOTAL CHANGE IN PE CAUSED BY MOVING FROM $\vec{r}_1$ to $\vec{r}_2$ IS

$$\Delta W_{21} = \int dW = -q_0 \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$$

(C) ELECTRIC POTENTIAL

$$\Delta V_{21} \equiv \frac{\Delta W_{21}}{q_0} = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r} \qquad [\Delta V] = \left[\frac{\text{Energy}}{\text{Charge}}\right] \doteq \left(\frac{\text{Joules}}{\text{Coulomb}}\right)$$

- NOTE THAT ELECTRIC POTENTIAL IS POTENTIAL ENERGY PER UNIT CHARGE

$$\Delta W_{21} = q_0 \Delta V_{21}$$

PE AND POTENTIAL ARE DIFFERENT THINGS! BUT THEIR NAMES OFTEN CAUSE CONFUSION.

(D) CONSERVATIVE FIELDS

$$\Delta V_{21} = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$$

$$dV = -\vec{E}(\vec{r}) \cdot d\vec{r} = -(E^x \hat{x} + E^y \hat{y} + E^z \hat{z}) \cdot (\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz)$$

$$dV = -(E^x dx + E^y dy + E^z dz)$$

$$dV(x,y,z) = -\left(E^x(x,y,z)dx + E^y(x,y,z)dy + E^z(x,y,z)dz \right)$$

but

$$dV(x,y,z) = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

($\frac{\partial V}{\partial x}$ means, for example, take the derivative of $V(x,y,z)$ w.r.t. x, leaving y and z fixed)

- So we identify

$$E^x = -\frac{\partial V}{\partial x}, \quad E^y = -\frac{\partial V}{\partial y}, \quad E^z = -\frac{\partial V}{\partial z}$$

A SHORTHAND WAY OF DOING THIS IS WITH THE VECTOR OPERATOR

$$\overline{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \qquad \text{(GRADIENT)}$$

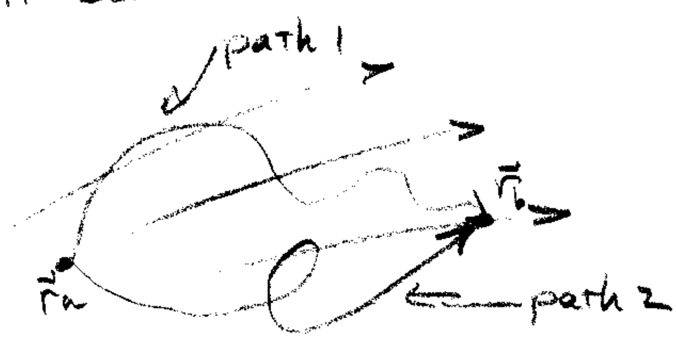
$$\vec{E} = -\overline{\nabla} V$$

- FIELDS THAT CAN BE EXPRESSED AS THE GRADIENT OF SOME SCALAR POTENTIAL ARE CALLED CONSERVATIVE

- NICE BECAUSE

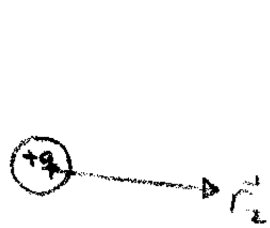
$$\Delta V_{21} = \int_{\vec{r}_1}^{\vec{r}_2} dV = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{r} = V(\vec{r}_2) - V(\vec{r}_1)$$

THE INTEGRAL DEPENDS ONLY ON THE ENDPOINTS $\vec{r}_1$ AND $\vec{r}_2$ – IT DOESN'T MATTER WHAT CURVE IS FOLLOWED!



Ex: POTENTIAL OF A POINT CHARGE

$$\Delta V_{21} = V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$$



FROM GAUSS' LAW (OR COULOMB'S LAW) WE HAVE

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \qquad d\vec{r} = dr\,\hat{r}$$

$$V(\vec{r}_2) - V(\vec{r}_1) = -\frac{q}{4\pi\epsilon_0} \int_{\vec{r}_1}^{\vec{r}_2} \frac{dr}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r} \Big|_{\vec{r}_1}^{\vec{r}_2}$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

LET $r_1 \to \infty$, AND SET $V(\infty) = 0$

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{r}}$$

POTENTIAL OF A POINT CHARGE WITH $V(\infty) \equiv 0$

Take the GRADIENT:

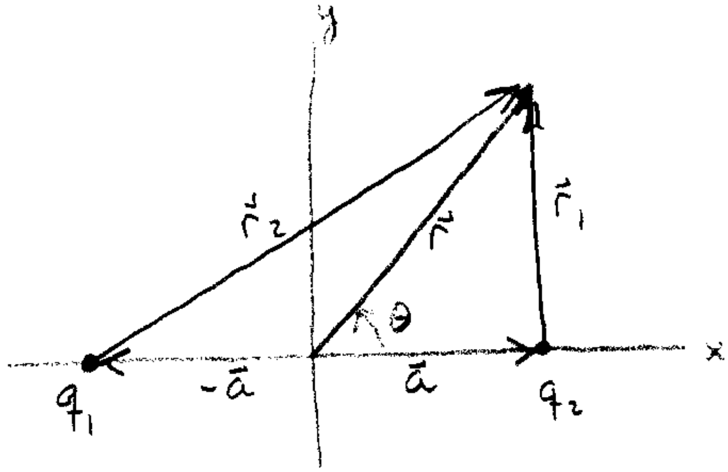
$$\vec{E} = -\vec{\nabla} V$$

$$= -\hat{r} \frac{\partial}{\partial r} \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r} \right)$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \quad \checkmark$$

RECOVER $\vec{E}$ FROM V by taking $-\vec{\nabla} V$

$\boxed{\text{Ex:}}$ POTENTIAL OF TWO POINT CHARGES



$$\vec{r}_1 = \vec{r} - \vec{a} \qquad \vec{r}_2 = \vec{r} - (-\vec{a}) = \vec{r} + \vec{a}$$

$$V_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r_1} = \frac{q}{4\pi\epsilon_0|\vec{r}-\vec{a}|} \qquad V_2 = \frac{q}{4\pi\epsilon_0 r_2} = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}+\vec{a}|}$$

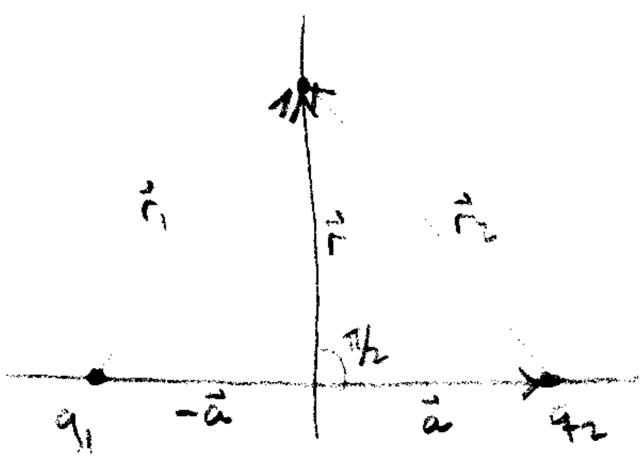
$$|\vec{r}-\vec{a}| = (r^2+a^2-2\vec{r}\cdot\vec{a})^{1/2} \qquad |\vec{r}+\vec{a}| = (r^2+a^2+2\vec{r}\cdot\vec{a})^{1/2}$$

$$\vec{r}\cdot\vec{a} = ra\cos\theta$$

- $V(\vec{r})$ is just $V(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r})$

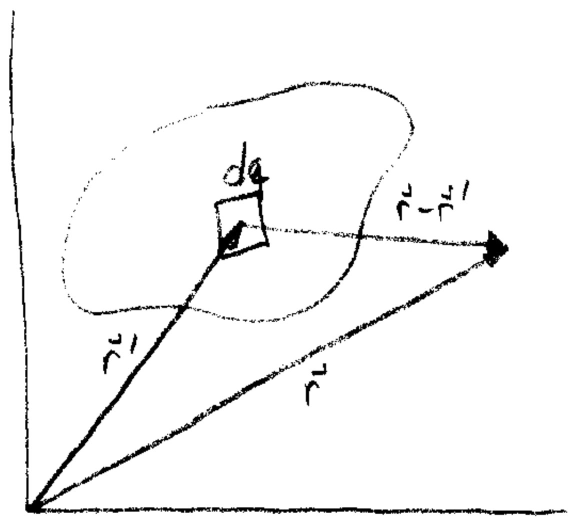
$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{(r^2+a^2-2ra\cos\theta)^{1/2}} + \frac{q_2}{(r^2+a^2+2ra\cos\theta)^{1/2}}\right]}$$

- a PARTICULAR CASE IS $\theta = \pi/2$



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{(r^2+a^2)^{1/2}} + \frac{q_2}{(r^2+a^2)^{1/2}}\right]$$

(E) POTENTIALS OF CONTINUOUS CHARGE DISTRIBUTIONS



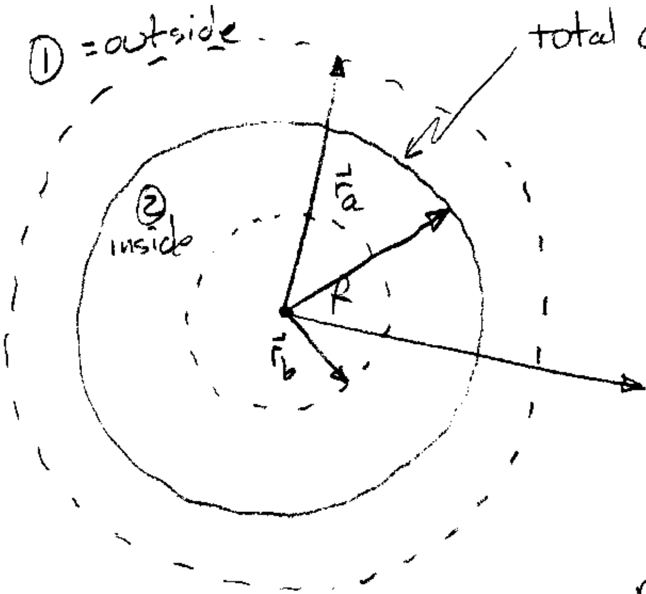
$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{dq(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$dq(\vec{r}') = \rho(\vec{r}')\, d\tau'$$

charge density ("ρ") — "volume" element ("$d\tau'$")

$$V(\vec{r}) = \int dV(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')\, d\tau'}{|\vec{r}-\vec{r}'|}$$

$\boxed{\text{Ex:}}$ SOLID SPHERICAL CONDUCTOR



$$\Delta V_{21} = V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r}$$

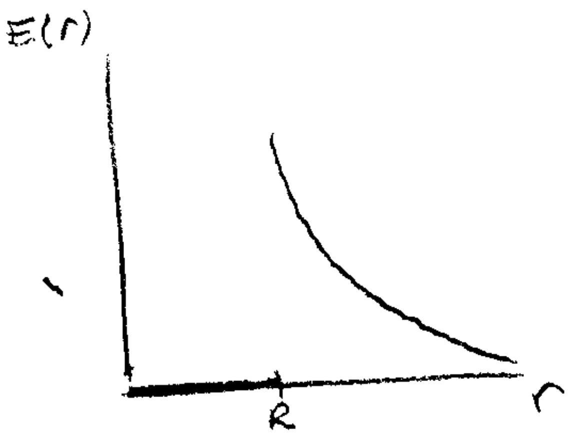
HAVE

Region ①: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$ for $R \le r < \infty$

Region ②: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$ for $0 \le r \le R$

FOR $\vec{r} = \vec{r}_a \ge R$:

$$V(\vec{r}_a) - V(\infty) = -\int_{\infty}^{\vec{r}_a} \vec{E} \cdot d\vec{r} = -\int_{\infty}^{\vec{r}_a} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr$$



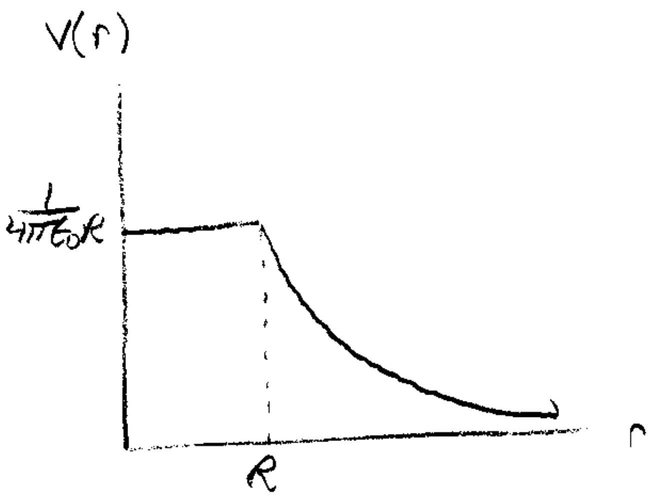
$$V(\vec{r}_a) = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{r_a} \right) \qquad \text{with } V_\infty \equiv 0$$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \qquad r \ge R}$$

For $\vec{r} = \vec{r}_b$

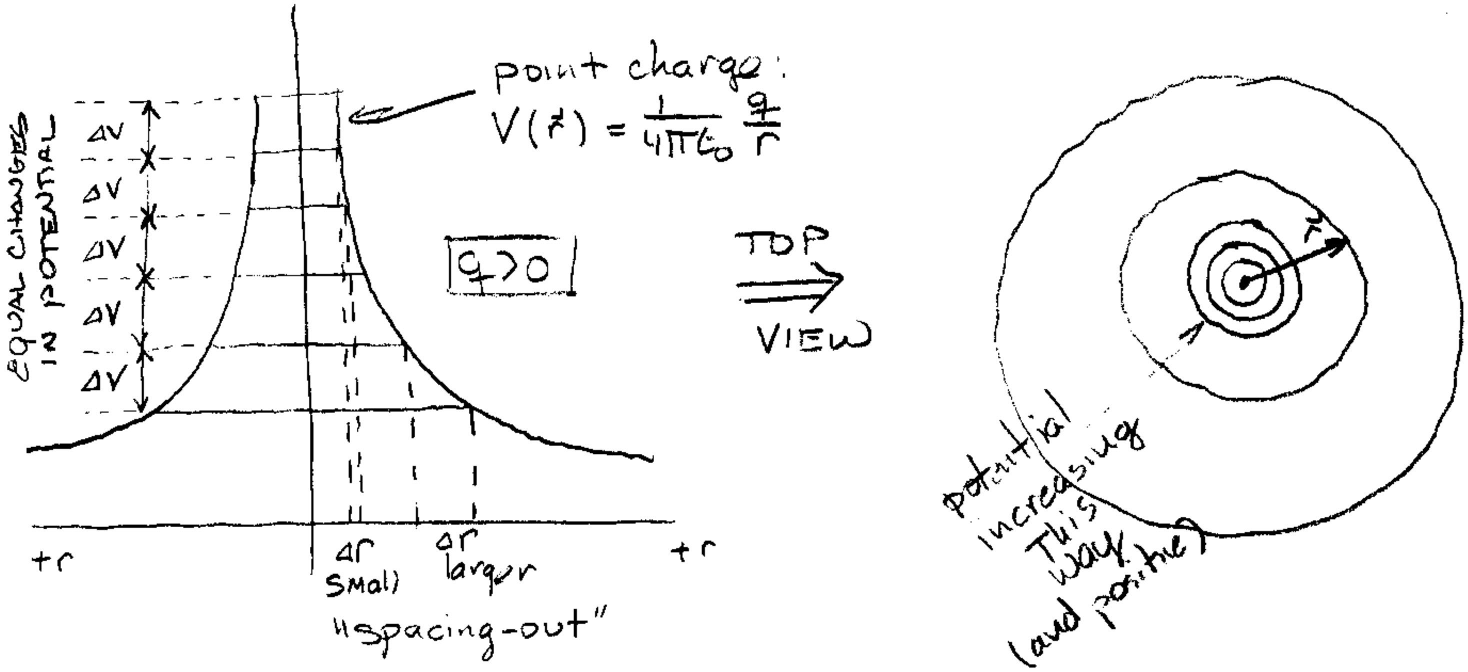
$0 \le \vec{r}_b \le R$

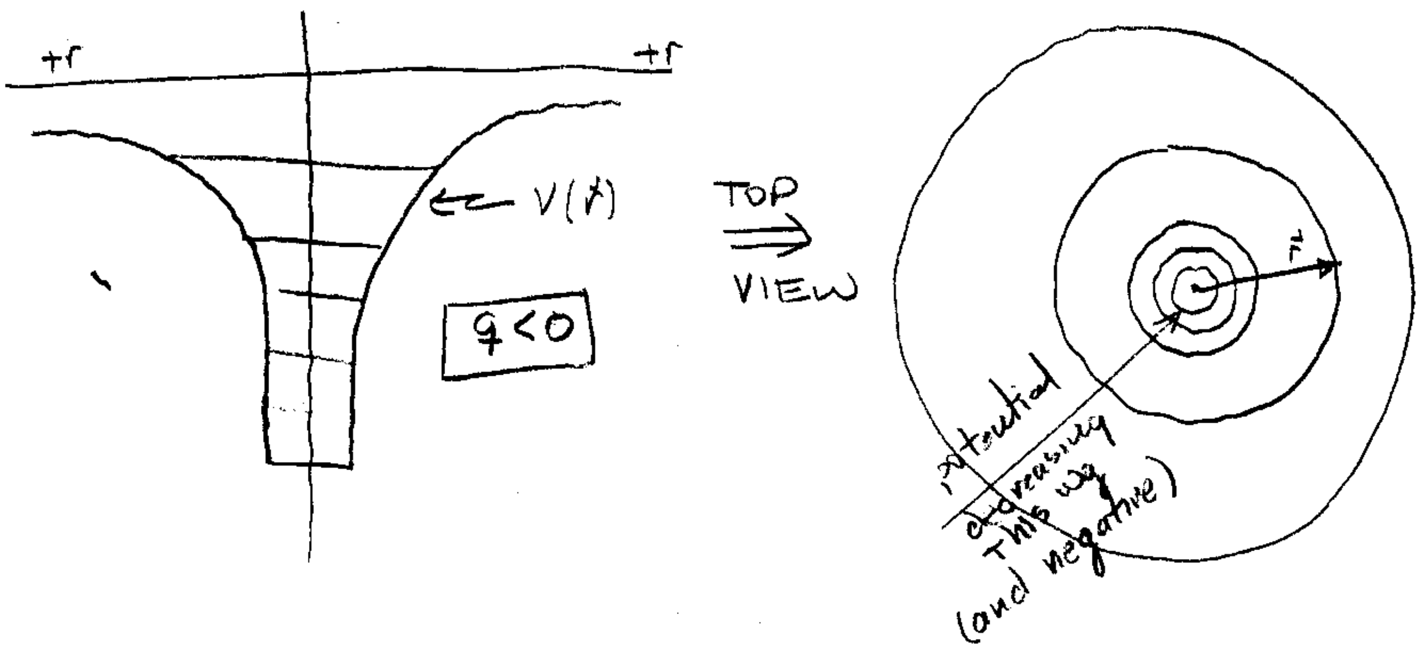
$$V(\vec{r}_b) - V(\infty) = -\int_{\infty}^{\vec{r}_b} \vec{E} \cdot d\vec{r} = -\int_{\infty}^{R} \vec{E} \cdot d\vec{r} - \int_{R}^{\vec{r}_a} \underbrace{(0)}_{\vec{E}=0 \text{ for } \vec{r} < R} \cdot d\vec{r}$$

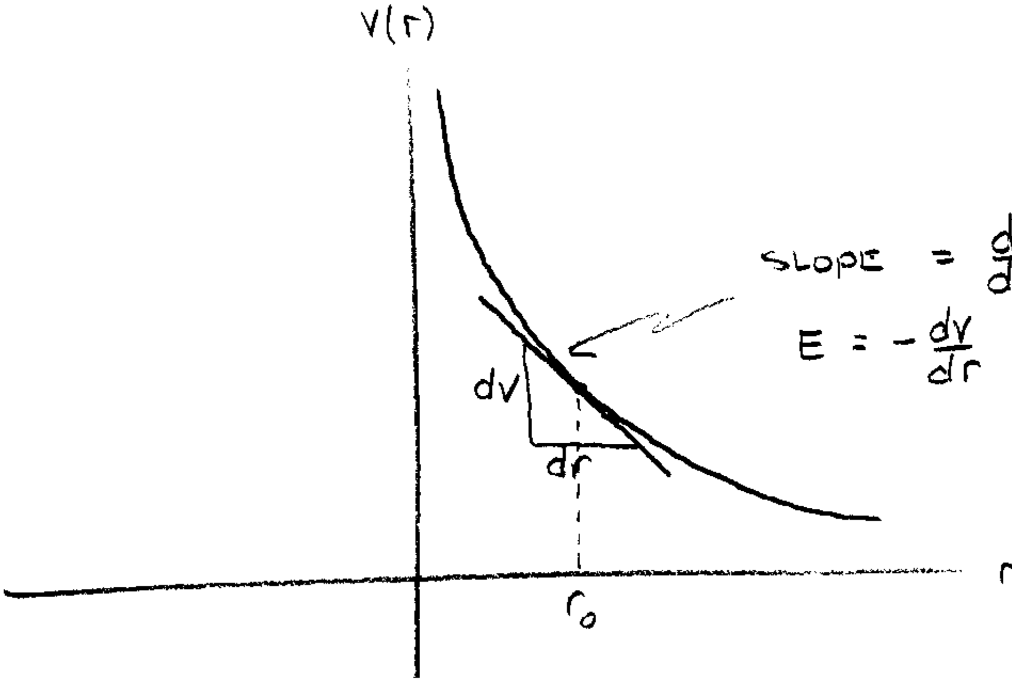


$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0 R} = \text{const, for all } r \le R}$$

EQUIPOTENTIALS



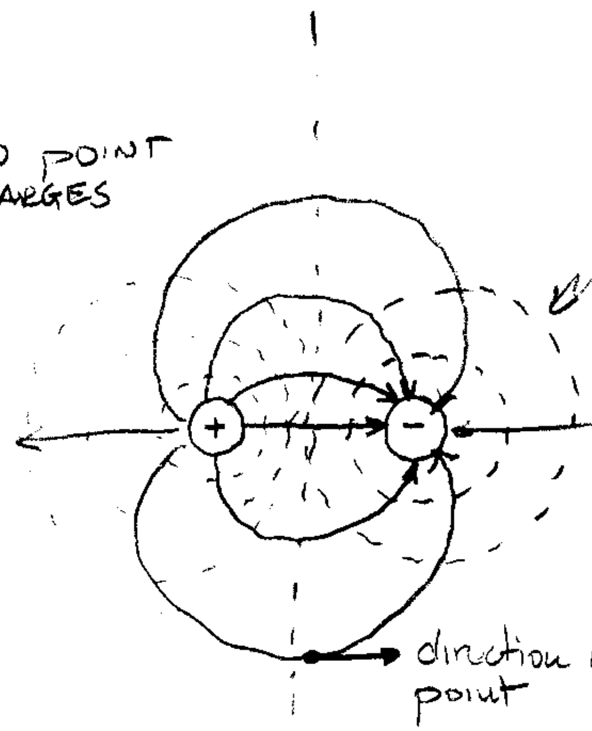




SLOPE $= \frac{dV}{dr} < 0$

$E = -\frac{dV}{dr} > 0 \Rightarrow$ E-field in direction of increasing r

TWO POINT CHARGES



equipotentials $\perp$ to $\vec{E}$-field-lines

direction of $\vec{E}$ at this point

IN GENERAL,