ELECTROSTATICS

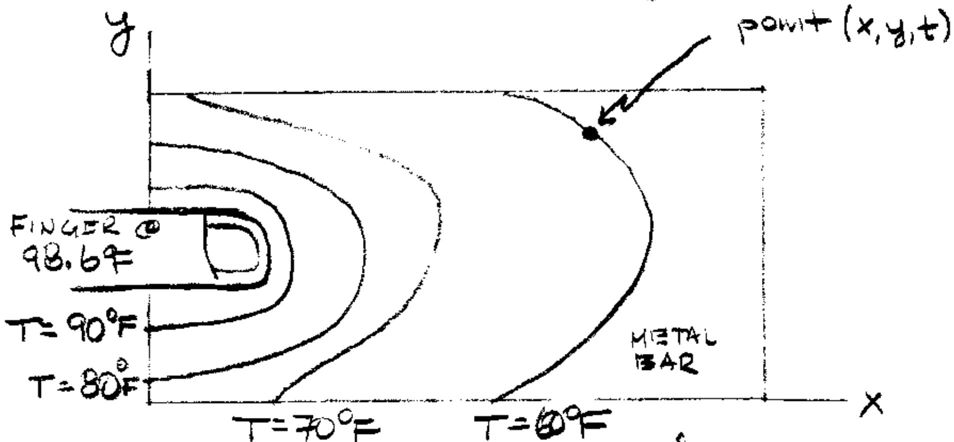
I. SCALAR AND VECTOR FIELDS

- A FIELD IS THE TOTALITY OF ALL OBJECTS DESCRIBING SOME QUANTITY AS A FUNCTION OF SOME OTHER QUANTITY.

- SCALAR FIELDS:

  Scalar FIELDS ARE OBJECTS WHICH HAVE MAGNITUDE ONLY.

$\boxed{Ex:}$ Temperature at some point (x,y,t) of a metal bar heated by a finger



Isotherms (lines of T=const.) on a metal bar heated by a finger as a function of position and time $T(x,y,t) = \text{const}$

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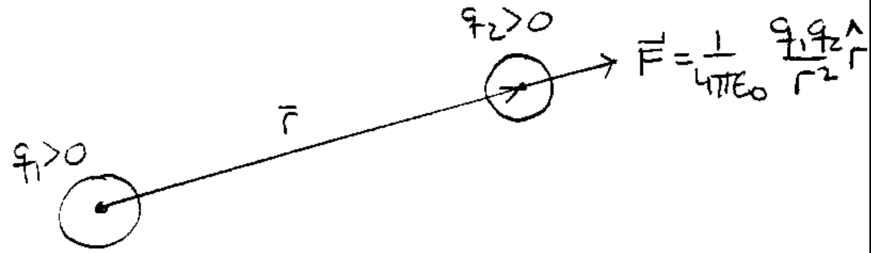
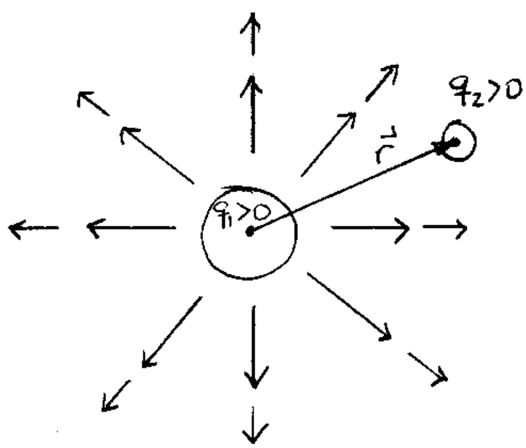
- VECTOR FIELDS

VECTOR FIELDS HAVE BOTH MAGNITUDE <u>AND</u> DIRECTION.

$\boxed{\text{EX:}}$ Force Field of a charge $q_1 > 0$ acting on a charge $q_2 > 0$.





Shows both the magnitude and direction of force of q_1 on q_2 as a function of q_2's position

$$\vec{F}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

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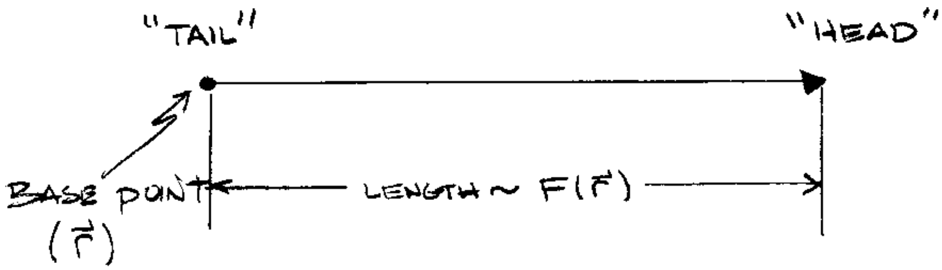
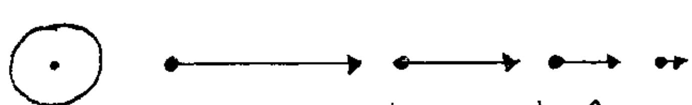
- Graphically, we indicate a vector by associating its <u>tail</u> with a base-point (i.e, the location in space where it is evaluated), and drawing an arrow along its direction, with a length proportional to its magnitude:



A VECTOR

$$\vec{F}(\vec{r}) = |\vec{F}(\vec{r})|\,\hat{r} = F(\vec{r})\,\hat{r}$$

$\boxed{\text{Ex:}}$ For Coulomb's Law



$$\vec{F}(\vec{r}) \sim \frac{1}{r^2}\,\hat{r}$$

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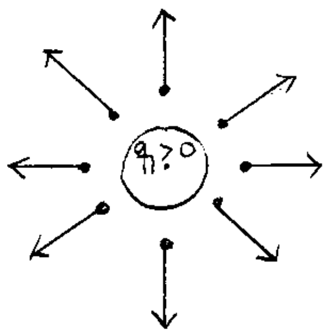
• $\boxed{\text{Ex:}}$ For $q_2 > 0$

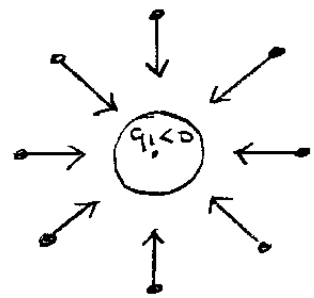
Field Lines around ⊕ charge



$$\vec{F}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

$$\vec{F} \sim \hat{r}$$

Field Lines around ⊖ charge



$$\vec{F}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

$$\vec{F} \sim -\hat{r}$$

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II. DEFINITION OF $\vec{E}$ - FIELD

$$\vec{F}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

re-write as

$$\vec{F}(\vec{r}) = q_2 \left(\frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \hat{r} \right)$$

Consider

$$\lim_{q_2 \to 0} \frac{\vec{F}(\vec{r})}{q_2} = \lim_{q_2 \to 0} \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \hat{r}$$
$$\equiv \vec{E}(\vec{r})$$

so

$$\vec{F}(\vec{r}) = q_2 \vec{E}(\vec{r})$$

$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$ $\Rightarrow$ Electric Field produced by a source charge q located at $r=0$ as a function of $\vec{r}$.

$\vec{F} = q_0 \vec{E} = q_0 \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \right) \Rightarrow$ Force on q_0 due to Electric Field produced by charge q.

- THIS IS A VERY GENERAL IDEA:

If we can compute $\vec{E}$ for <u>any</u> charge distribution, we can determine the force produced by it on some charge $q_0 \rightarrow \quad \vec{F} = q_0 \vec{E}$

ELECTROSTATICS

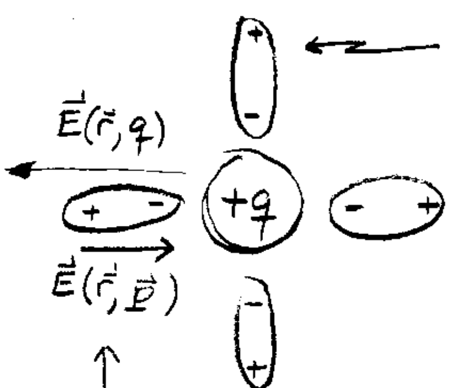
III. $\vec{E}$-FIELDS IN DIELECTRIC MEDIA

- POLARIZATION

SUPPOSE WE WANT TO CALCULATE THE ELECTRIC FIELD OF A POINT CHARGE INSIDE OF A MACROSCOPIC MATERIAL, SUCH AS HUMAN FLESH, RATHER THAN THE VACUUM. HOW IS COULOMB'S LAW MODIFIED?

TURNS OUT that THE LAW REMAINS THE SAME, ONLY ϵ_0 CHANGES TO SOME ϵ that DEPENDS ON THE MATERIAL. THIS IS DUE TO POLARIZATION

⊕ charges in surrounding medium get pushed-away, ⊖ charges get pulled closer.



Polarized medium sets-up $\vec{E}$-fields that oppose $\vec{E}$-field that caused them. E_{total} is smaller than in vacuum.

- Useful to consider $K_e \equiv \epsilon/\epsilon_0$

K_e vacuum $= 1.0000$

K_e air $= 1.00054$

K_e flesh $= 8$ (avg).

$\epsilon = K_e \epsilon_0$

$$\vec{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{r}$$

$$\vec{E} = \frac{1}{4\pi K_e \epsilon_0} \frac{q}{r^2} \hat{r}$$

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IV. $\vec{E}$-FIELD LINES

- IN GENERAL
  - $\vec{E}$-FIELD LINES INDICATE THE DIRECTION OF THE $\vec{E}$-FIELD AT ANY POINT IN SPACE.
  - THE LOCAL DENSITY (FIELD LINES/UNIT AREA) INDICATE THE STRENGTH OF THE $\vec{E}$-FIELD (ITS MAGNITUDE) IN THAT REGION
  - FIELD LINES ALWAYS BEGIN AND END ON CHARGE SOMEWHERE. BY CONVENTION, THEY BEGIN ON ⊕ CHARGE AND END ON ⊖ CHARGE.

    (Question: where do the field lines of a single ⊕ charge end?)
  - Field Lines never cross (ambiguous.)

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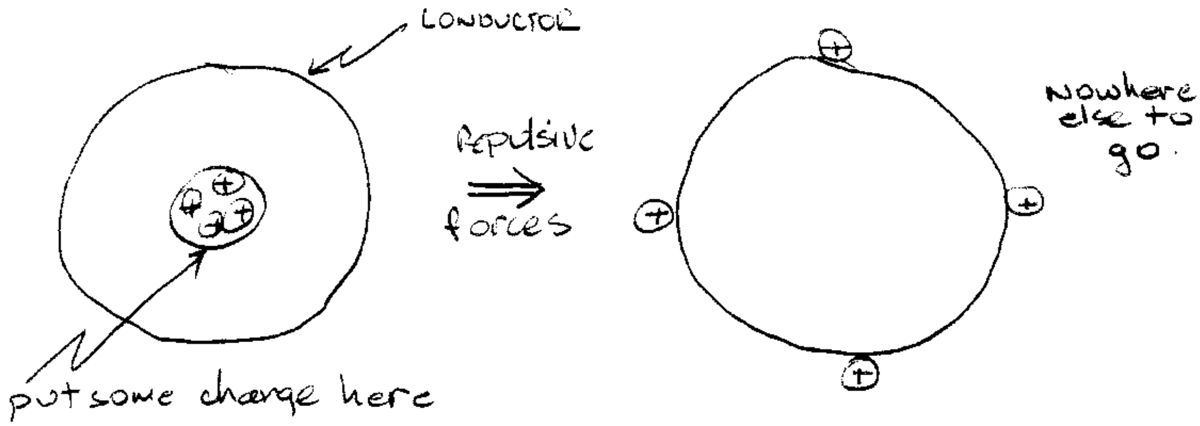
- FIELD LINES AND CONDUCTORS

We will soon show that all of the FREE CHARGE on a conductor resides entirely on the surface of the conductor.

The idea is that free charge is not bound to any particular atom in the conductor — it is free to move around. Like charges repel, so they will exert forces on each other that will cause them to move away from each other. They will get as far away as they can (the surface) then distribute themselves as far from each other as is possible.



- Electric field lines are everywhere perpendicular to surface of conductor
    - If not, then they would experience forces causing them to move until so.

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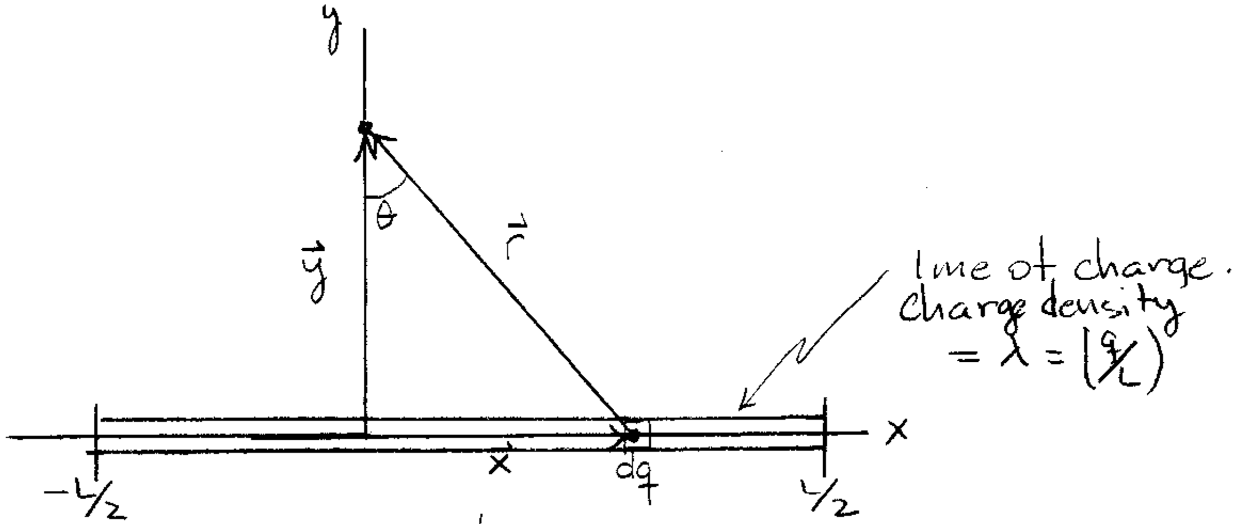
V. $\vec{E}$-FIELDS OF CONTINUOUS CHARGE DISTRIBUTIONS



$$d\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}$$

$$\vec{r} = x\hat{x} + y\hat{y}$$

$$r^2 = \vec{r}\cdot\vec{r} = x^2+y^2$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$$

$$dq = (q/L)dx = \lambda dx \qquad \lambda \equiv (q/L)$$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{(x^2+y^2)}\left(\frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}\right)$$

$\hat{x}$-component vanishes by symmetry.

$$d\vec{E} = \left(\frac{1}{4\pi\epsilon_0}\frac{\lambda y\, dx}{(x^2+y^2)^{3/2}}\right)\hat{y}$$

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$$\frac{dx}{(x^2+y^2)^{3/2}} = \frac{1}{y^2}\,\frac{d(x/y)\,y}{(1+(x/y)^2)^{3/2}}$$

$$= \frac{1}{y^2}\,\frac{du}{(1+u^2)^{3/2}} \qquad u \equiv x/y$$

let $u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$1+u^2 = 1+\tan^2\theta = \sec^2\theta$

$$\sqrt{1+u^2} = (1+u^2)^{3/2} = (1+\tan^2\theta)^{3/2} = \sec^3\theta$$

$$\frac{dx}{(x^2+y^2)^{3/2}} = \frac{1}{y^2}\,\frac{\sec^2\theta\, d\theta}{\sec^3\theta}$$

$$= \frac{1}{y^2}\cos\theta\, d\theta$$

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+y^2)^{3/2}} = \frac{1}{y^2}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta$$

$$= \frac{1}{y^2}\sin\theta\Big|_{-\pi/2}^{\pi/2}$$

$$= 2/y^2$$

$$\vec{E} = \int d\vec{E} = \frac{\lambda y}{4\pi\epsilon_0}\cdot\frac{2}{y^2}\cdot\hat{y}$$

$$\boxed{\vec{E}(\vec{y}) = \frac{\lambda}{2\pi\epsilon_0 y}\,\hat{y}}$$

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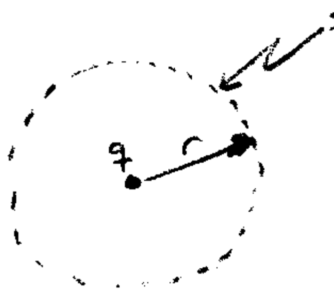
WOW! THAT WAS PAINFUL!

THERE'S GOT TO BE A BETTER WAY...

VI. GAUSS' LAW

- IMAGINE SURROUNDING A POINT-CHARGE WITH SOME SPHERE OF RADIUS r CENTERED ON THE CHARGE



NOTICE THAT

- AREA OF SPHERE $= 4\pi r^2 = A(r)$
- FIELD STRENGTH AT FIXED r IS

$$E(r) = \frac{1}{4\pi\epsilon_0} q/r^2$$

$$E(\vec{r})A(r) = \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)\cdot\left(4\pi r^2\right) = q/\epsilon_0$$

- NOTE THAT THIS QUANTITY IS INDEPENDENT OF THE RADIUS OF THE SPHERE.

- NOTE THAT $E(\vec{r}) \sim q$

IF WE ASSOCIATE THE # OF FIELD LINES CROSSING A UNIT AREA WITH THE AMOUNT OF CHARGE CONTAINED IN THE SPHERE, WE STILL HAVE

$$E(\vec{r})A(r) = q/\epsilon_0 \sim \text{\# of field lines crossing sphere.}$$

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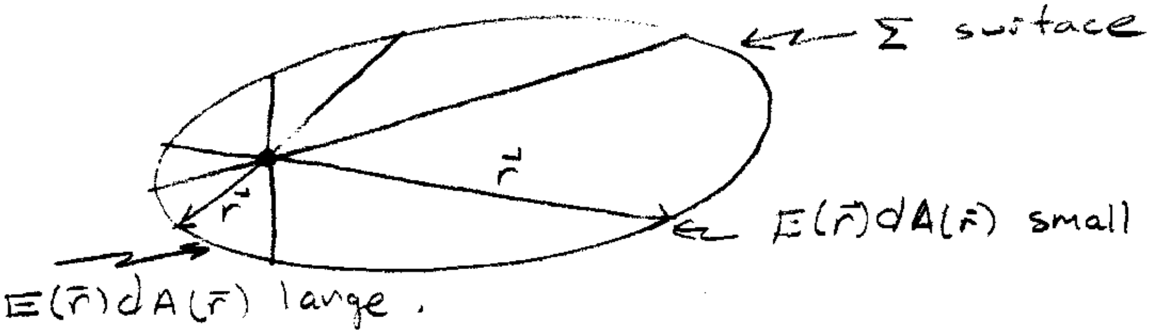
- NOW WE CAN change THE SHAPE OF THE SURFACE (SPHERE) WITHOUT CHANGING THE # OF FIELD LINES THAT CROSS IT



But still have:

$$q/\epsilon_0 = \oint_\Sigma \vec{E}(\vec{r})\cdot d\vec{A}(r) = \text{const.}$$

THIS IS GAUSS' LAW

$$\boxed{q_{encl} = \epsilon_0 \oint_\Sigma \vec{E}(\vec{r})\cdot d\vec{A}(\vec{r})}$$

q_{encl} = total charge enclosed inside of Σ.

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- THAT'S NICE, BUT HOW IS THIS USEFUL?
- SYMMETRY, SYMMETRY, SYMMETRY

Recall how we already used symmetry TO REASON THAT THE $\hat{x}$-COMPONENT OF THE INFINITE LINE CHARGE HAD TO BE ZERO (RATHER THAN CALCULATING IT.)

GAUSS' LAW GIVES US THE POWER TO USE SYMMETRY IN A MORE GENERAL WAY, ESPECIALLY IN CASES WITH:

- CYLINDRICAL SYMMETRY
- SPHERICAL SYMMETRY
- PLANAR SYMMETRY

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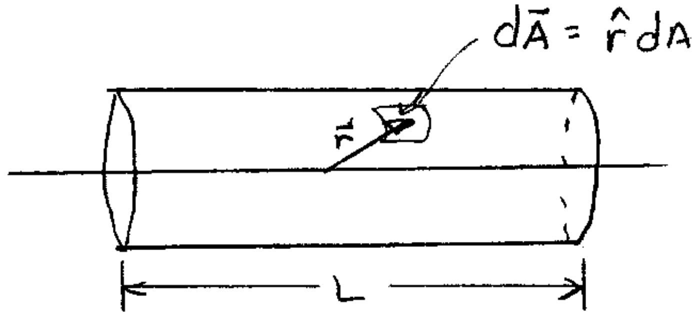
• $\boxed{Ex:}$ Cylindrical Symmetry – an infinite line-charge (revisited)



GAUSS' LAW: $q_{enc} = \epsilon_0 \oint_{\Sigma} \vec{E}(\vec{r}) \cdot d\vec{A}$

$$q_{enc} = \left(q/L\right) L = \lambda L$$

$$\vec{E}(\vec{r}) = E(r)\,\hat{r} \qquad d\vec{A} = \hat{r}\, dA$$

$$\vec{E}(\vec{r}) \cdot d\vec{A} = E(r)\, dA\, (\hat{r} \cdot \hat{r}) = E(r)\, dA$$

$$\lambda L = \epsilon_0 \oint_{\Sigma} E(r)\, dA$$

$$\lambda L = \epsilon_0\, E(r)\, (2\pi r) L$$

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$\boxed{\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}}$$

Much simpler to find than before.

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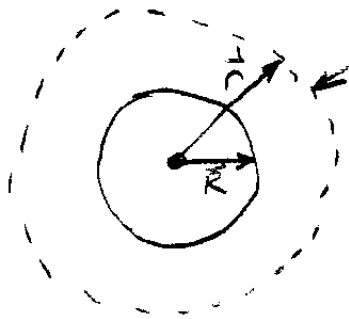
- SPHERICAL SYMMETRY — SPHERICAL SHELL OF CHARGE q.



Σ imaginary sphere surrounding spherical charge distribution

$$q_{encl} = \epsilon_0 \oint_{\Sigma} \vec{E}(\vec{r}) \cdot d\vec{A}(\vec{r})$$

$$\vec{E}(\vec{r}) = E(r)\,\hat{r} \qquad dA = \hat{r}\, dA$$

$$q_{encl} = \epsilon_0 E(r) \oint_{\Sigma} dA$$

$$q_{encl} = \epsilon_0 E(r)\,(4\pi r^2)$$

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2}$$

$$\boxed{\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}}$$

SAME AS $\vec{E}$-FIELD OF POINT CHARGE q located at $r=0$.

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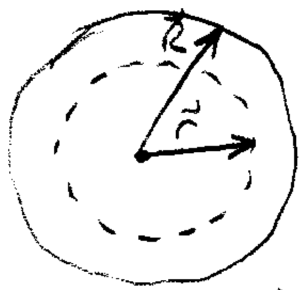
- BONUS: FIELD INSIDE OF THE SPHERICAL CHARGE DISTRIBUTION



$$q_{encl} = \epsilon_0 \oint_{\Sigma} \vec{E}(\vec{r}) \cdot d\vec{A}(\vec{r})$$

But $q_{encl} = 0$ (remember why?)

$$0 = \epsilon_0 E(r)(4\pi r^2)$$

$\Rightarrow \boxed{E(r) = 0} \Rightarrow$ Electric Field inside of a conductor is zero.

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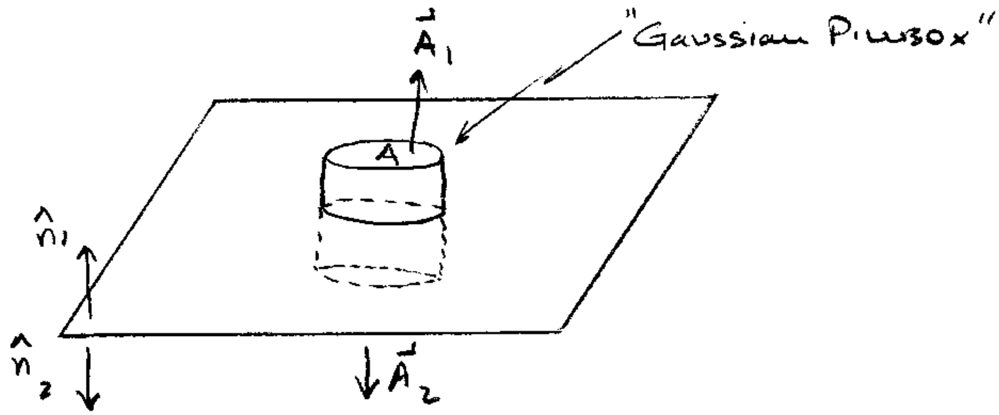
- $\boxed{\text{Ex:}}$ PLANAR SYMMETRY — $\vec{E}$-FIELD OF AN INFINITE PLANE CHARGE



$$q_{encl} = \epsilon_0 \oint_{\Sigma} \vec{E}(\vec{r}) \cdot d\vec{A}(\vec{r})$$

$\vec{E}(\vec{r})$ has no components $\parallel$ to surface

$$q_{encl} = \epsilon_0 \left(\vec{E}_1 \cdot \vec{A}_1 + \vec{E}_2 \cdot \vec{A}_2 \right)$$

$$= 2 \epsilon_0 E A$$

$$q_{encl} = \left(\frac{q}{A} \right) A = \sigma A \qquad \sigma \equiv \text{surface charge density} \equiv \left(\frac{q}{A} \right)$$

$$\sigma A = 2 \epsilon_0 E A$$

$$\boxed{\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}}$$

Note that this is independent of how far from the plane one is.
(Why?)