VECTORS

The concept of vectors is of particular importance to physicists. A vector is a quantity that has both magnitude and direction. This is as opposed to scalar quantities which have only a magnitude.

e.g.:

Vectors

- velocity = 60 mph south
- displacement = distance moved (it makes a difference if a punter kicks a football 100 yds st. up or 100 yds down field)
- force (may it be with you – in your direction – rather than against you)

Scalars

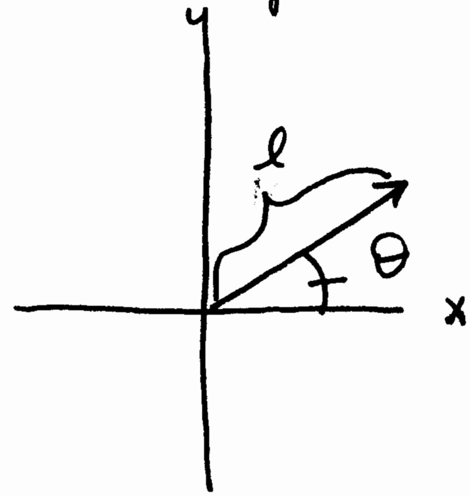
- Speed = 60 mph (the cop doesn't care which direction)
- Temperature (°)
- Time (h m s)
- Relative Humidity (%)
- Interest rate
- energy

We will encounter a large no. of vector quantities in this course and it will be impt. to know how to treat them.

We denote vectors by BOLDFACE type (as the text does) or by an arrow (as we'll do in

e.g. **V** or $\vec{V}$

In two dimensions we require two numbers to specify a vector. Intuitively these are a magnitude and a direction = <u>components</u> (length $= \ell$) (angle $= \theta$)



Walk 45 paces NE to the treasure

$\ell = 45$ paces ≈ 45 m

$\theta = 45^\circ$

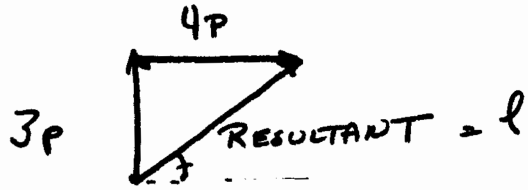
(We note parenthetically that only for displacement [distance] is this graphical representation absolutely true; everything else is a scaled representation (e.g. 1 cm on the paper represents 10 km s^{-1}, etc)

Given a ruler and a protractor one may add vectors graphically:

Long John Silver walks 3 paces north

$\uparrow$ then 4 paces east $\longrightarrow$

to find the final displacement (resultant) of Long John Silver



If we measure we will find

$l = 5$ paces $\quad \theta = 37°$

We can take any arbitrary no of vectors (e.g. 4 paces NNW, 3 paces N, 2 paces ENE) and add them together graphically by placing the "foot" of one vector at the "head" of the previous vector to find the resultant or "head tail distance"

But this is a cumbersome process.

We may, however, specify vectors by different components in Cartesian Coordinate System:

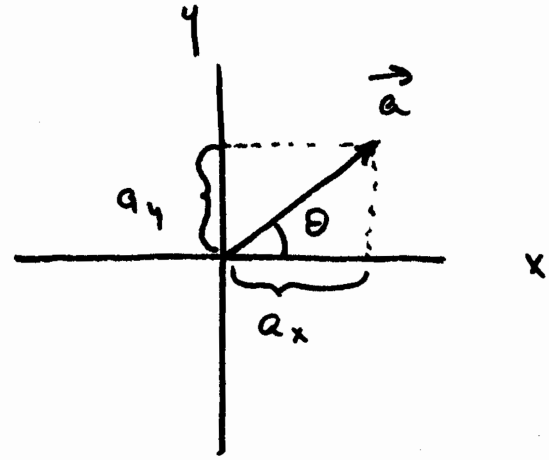
$$\text{Instead of } \vec{a} = (l, \theta)$$

$$\text{we use } \vec{a} = (a_x, a_y)$$

Trig def'ns: 

$\sin\theta = \frac{y}{r}$ $\tan\theta = \frac{y}{x}$

$\cos\theta = \frac{x}{r}$



$$a_x = |a| \cos\theta$$

$$a_y = |a| \sin\theta$$

In the above, simple example we have

$$\vec{a}_1 = (0,\ 3 \text{ paces})$$
$$\vec{a}_2 = (4 \text{ paces},\ 0)$$

To add these vectors we simply add the components separately

$$a_x = a_{1x} + a_{2x} \quad (= 0 + 4)$$
$$a_y = a_{1y} + a_{2y} \quad (= 3 + 0)$$

Thus $\vec{a} = (4 \text{ paces}, 3 \text{ paces})$

In general

$$a_x = \sum_i a_{ix}$$
$$a_y = \sum_i a_{iy}$$

for any number of vectors

Recalling our geometry

This denotes the magnitude $\longrightarrow |a|^2 = a_x^2 + a_y^2 = 4^2 + 3^2 = 25$

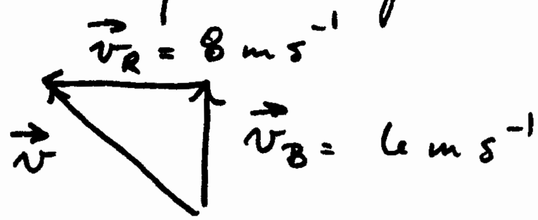
$$|a| = 5 \text{ paces}$$

Relating to the angle

$$\tan\theta = \frac{a_y}{} = 0.75 \qquad \theta = \tan^{-1} 0.75 = 37^\circ$$

For most applications in this course we'll find the cartesian notation most useful. Now that we have the basics let's try a few games:

(1) A boat which can travel 6 m s^{-1} heads north across a rapid section of the Colorado which is traveling west at 8 m s^{-1}; what is the resultant velocity (magnitude + direction)



$$|v|^2 = 6^2 + 8^2 = 36 + 64 = 100$$

$$|v| = 10\ m\ s^{-1}$$

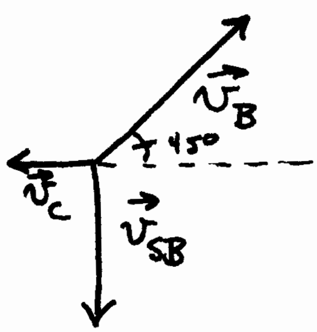
$$\vec{v} = \vec{v}_B + \vec{v}_R$$

$$= (0, 6\ m\ s^{-1}) + (-8\ m\ s^{-1}, 0)$$

$$\vec{v} = (v_x, v_y) = (-8\ m\ s^{-1}, 6\ m\ s^{-1})$$

But armed with these tools we can tackle more difficult problems:

Consider a boat headed NE at $10\ m\ s^{-1}$ with a current of $2\ m\ s^{-1}$ W. On the boat is a skateboarder who is moving $4\ m\ s^{-1}$ S (wrt the boat). What is his velocity relative to the shore. We have then



$$\vec{v}_B = (10\,ms^{-1}\cos 45°,\ 10\,ms^{-1}\sin 45°)$$
$$= (7.07\,ms^{-1},\ 7.07\,ms^{-1})$$

$$\vec{v}_c = (-2\,ms^{-1},\ 0\,ms^{-1})$$

$$\vec{v}_{SB} = (0\,ms^{-1},\ -4\,ms^{-1})$$

$$v_x = 7.07\,ms^{-1} + (-2\,ms^{-1}) + 0\,ms^{-1}$$
$$= 5.07\,ms^{-1}$$

$$v_y = 7.07\,ms^{-1} + 0\,ms^{-1} + (-4\,ms^{-1})$$
$$= 3.07\,ms^{-1}$$

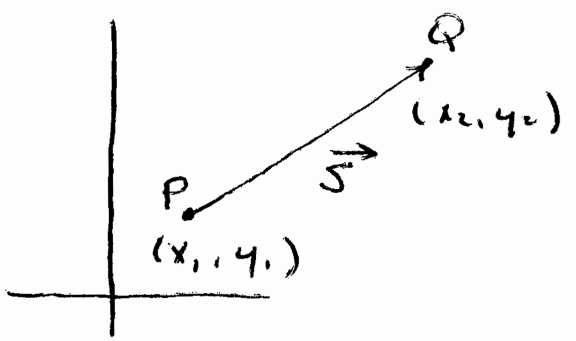
$$\vec{v} = (5.07\,ms^{-1},\ 3.07\,ms^{-1})$$

$$|v| = 6\,ms^{-1} \qquad \theta = \tan^{-1}\frac{v_y}{v_x} = 31°$$

MOTION IN A PLANE

Today we are going to begin to expand our treatment of motion from the one dimensional case to two dimensions. Much of what we discuss will be strangely familiar - we've covered most of it before in talking about rectilinear motion. Basically we have the same quantities displacement, velocity, acceleration, force. In 2d motion we must deal with the vector nature of these quantities

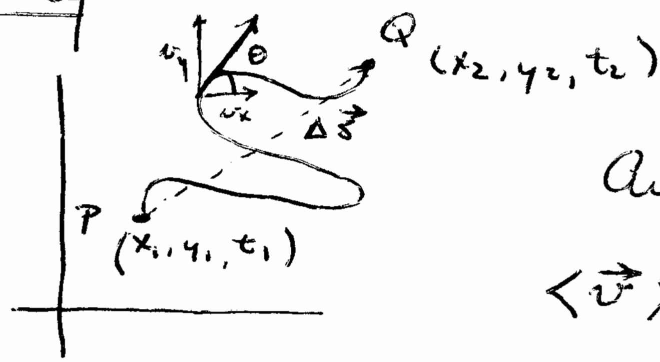
Displacement $\vec{s}$



$$|s| = \sqrt{(x_1-x_2)^2+(y_2-y_1)^2}$$

$\vec{s} = \vec{s}(t)$ Because s is a vector may treat components separately $x = x(t)$ $y = y(t)$

Velocity



Average velocity

$$\langle \vec{v} \rangle = \frac{\Delta \vec{s}}{\Delta t}$$

regardless of path from P – Q

Instantaneous velocity

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{s}}{\Delta t} = \frac{d\vec{s}}{dt}$$

Can treat components separately

$$\langle v_x \rangle = \frac{\Delta x}{\Delta t} \qquad \langle v_y \rangle = \frac{\Delta y}{\Delta t}$$

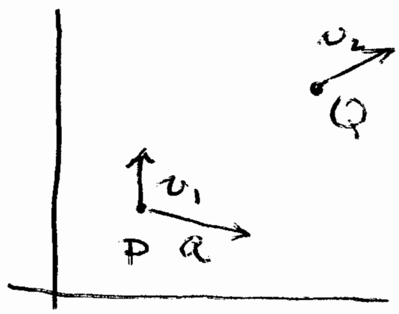
$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt}$$

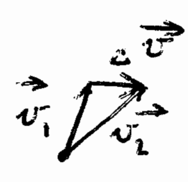
with

$$|v| = \sqrt{v_x^2 + v_y^2}$$

$$\tan\theta = \frac{v_y}{v_x}$$

Acceleration





$$\langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t}$$

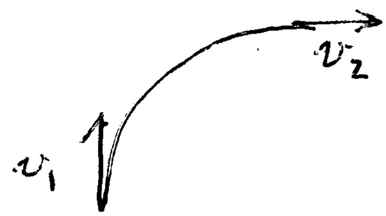
Instantaneous acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Again can divide into components

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{d^2y}{dt^2}$$

Consider a car which is traveling $10\ m\ s^{-1}$ around a curve in $10\ s$

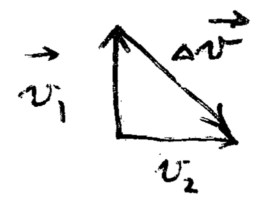


$v_1 = 10\ ms^{-1}$ north

$v_2 = 10\ ms^{-1}$ east

$v_{1x} = 0 \quad v_{1y} = 10\ ms^{-1}$

$v_{2x} = 10\ ms^{-1} \quad v_{2y} = 0$



$\Delta v_x = 10\ ms^{-1} \quad \Delta v_y = -10\ ms^{-1}$

$$\Delta v = \sqrt{v_x^2 + v_y^2} = \sqrt{(10)^2 + (-10)^2}$$

$$\Delta v = 14\ ms^{-1} \qquad \Delta t = 10\ s$$

$$\langle a \rangle = \frac{\Delta v}{\Delta t} = 1.4\ m\ s^{-2}$$

Is this a real acceleration - the speed of the car remains the same $10\ ms^{-1}$ and the direction has

Projectile Motion

FREELY FALLING "PARTICLE" acting under the influence of GRAVITY (ignore wind/air resistance) follows a path called a TRAJECTORY

@ $t = 0$ $\vec{v} = v_0$ at angle θ from ground

We separate motion into 2 components

(x)

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$[\ x = v_{0x} t\ ]$$

$$v_x = v_{0x} + a_x t$$

(y)

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$

$$[\ y = \frac{1}{2} g t^2 + v_{0y} t\ ]$$

$$v_y = v_{0y} + a_y t$$

The eqns above are completely general

[Eqns in brackets assume $x_0, y_0, a_x = 0$]

We separate motion into components and follow the motion independently

$v_{0x} = v_0 \cos\theta_0$

$a_x = 0$

$x_0 = 0$

$v_x = v_0 \cos\theta_0$

$x = v_0 \cos\theta_0 t$

$v_{0y} = v_0 \sin\theta_0$ $\qquad v_y = v_0 \sin\theta_0 - g\,t$

$a_y = +g = -9.8\,ms^{-2}$ $\qquad y = v_0 \sin\theta_0 t + \frac{1}{2} g t^2$

$y_0 = 0$

At any given time

$$v = \sqrt{v_x^2 + v_y^2} \quad @ \text{ angle } \theta$$

$\ni$ $\qquad \tan\theta = \dfrac{v_y}{v_x}$

If we wish to describe the motion in terms of a relationship between x and y (i.e. plot the trajectory in the above graph)

$$x = v_0 \cos\theta_0 t \Rightarrow t = \frac{x}{v_0 \cos\theta_0}$$

Subst in eqns for y

$$y = \frac{\cancel{v_0} \sin\theta_0 \cdot x}{\cancel{v_0} \cos\theta_0} + \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

$$y = \tan\theta_0 \, x + \frac{g}{2 v_0^2 \cos^2\theta_0} x^2$$

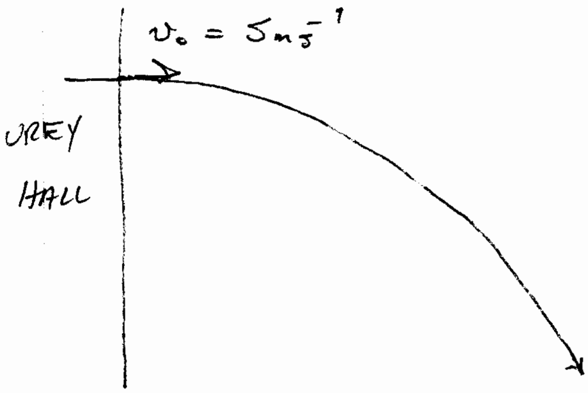
This eqn has form

$$y = ax + bx^2$$

where $a = \tan\theta_0$ $\qquad b = \dfrac{g}{2 v_0^2 \cos\theta_0^2}$

which is eqn for a parabola

WATERMELON



Remember question posed at end of last week — how do we treat watermelon tossed horizontally

How long to fall; what is final $\vec{v}$?

2D — separate vectors

X

INITIAL

$X_0 = 0$

$v_{0x} = 5\,ms^{-1}$

$a_x = 0$

FINAL

$x = \ ?$

$v_x = 5\,ms^{-1}$ Since $a_x = 0$, $v_x = $ const

$t = \ ?$ ⟵ same value of t ⟶

Y

$y_0 = 15\,m$

$v_{0y} = 0$

$a_y = g = -9.8\,ms^{-2}$

$y = 0$

$v_y = \ ?$

$t = \ ?$

$$x = \cancelto{0}{x_0} + v_{0x}t + \tfrac{1}{2}\cancelto{0}{a_x}t^2$$

$$x = v_{0x}t = 5\,ms^{-1}\,t$$

$$v_x = v_{0x} + \cancelto{0}{a_x t}$$

$$v_x = 5\,ms^{-1}$$

$$y = y_0 + \cancelto{0}{v_{0y}t} + \tfrac{1}{2}a_y t^2$$

$$y = 15\,m + \tfrac{1}{2}(-9.8\,ms^{-2})t^2$$

$$= 15\,m - 4.9\,ms^{-2}\,t^2$$

$$v_y = v_{0y} + a_y t$$

How long does it take to fall. Eqn for y

$$0 = 15\,m - 4.9\,ms^{-2}t^2$$

$$t^2 = \frac{15m}{4.9\,ms^{-2}}$$

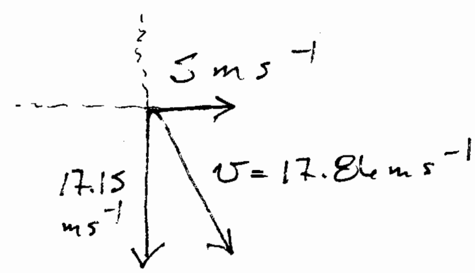
$$\underline{t = 1.75\,s}$$

How fast is it going?

$$v_y = -9.8\,ms^{-2} \cdot t = -9.8\,ms^{-2} \cdot 1.75\,s$$

$$= -17.15\,ms^{-1}$$

$$v_x = 5\,ms^{-1}$$

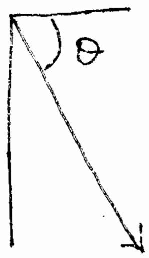


$$|v| = \sqrt{v_x^2 + v_y^2} = \sqrt{(5ms^{-1})^2 + (17.15ms^{-1})^2}$$

$$= 17.86\,ms^{-1}$$

(Note this is the same magnitude as when the melon was tossed st-up - Could there be some conservation principal here?)

Angle of impact



$$\tan\theta = -\frac{17.15}{5} = -3.43$$

$$\theta = -74°$$

How far from foot of the wall

$$x = x_0 + v_{0x} t$$

$$= 5\,m\,s^{-1} \cdot 1.75\,s$$

$$= 8.75\,m$$ called the range

More Projectile Motion

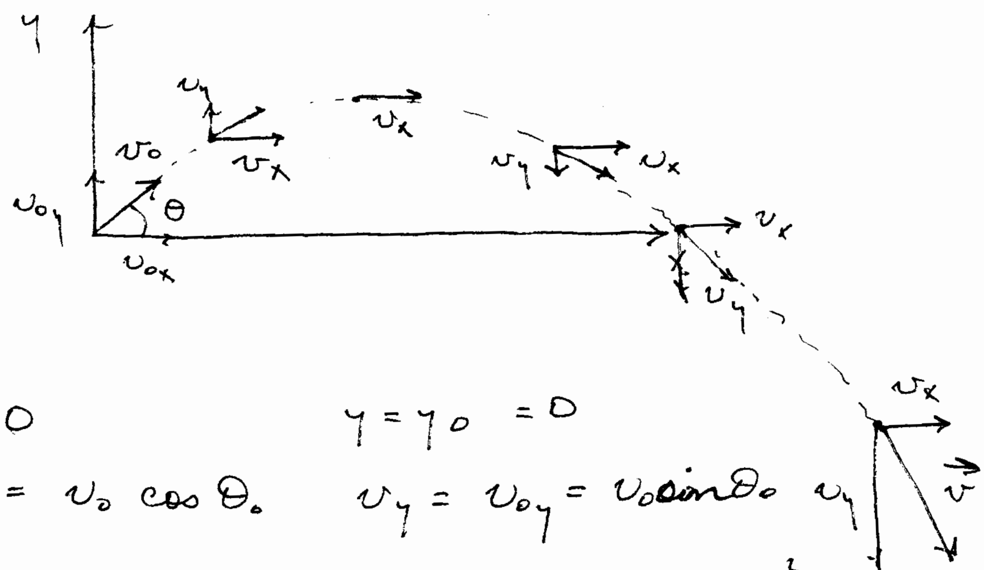
Path followed = trajectory (athletic sports ...)

Assume const g, neglect air resistance (flat, non-rotating earth in vacuum!)

x - axis horizontal $a_x = 0$

y - axis vertical $a_y = g = -9.8\,ms^{-2}$

Take the case $x = 0$ $y = 0$ $t = 0$



$t = 0$

$x = x_0 = 0$ $\qquad$ $y = y_0 = 0$

$v_x = v_{0x} = v_0 \cos\theta_0$ $\qquad$ $v_y = v_{0y} = v_0 \sin\theta_0$

$a_x = 0$ $\qquad$ $a_y = g = -9.8\,ms^{-2}$

$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$

$\underline{x = v_0 \cos\theta_0 t}$

$v_x = v_{0x} + a_x t$

$\underline{v_x = v_0 \cos\theta_0}$

$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$

$\underline{y = v_0 \sin\theta_0 t + \frac{g}{2} t^2}$

$\underline{v_y = v_0 \sin\theta_0 + g t}$

The RANGE R

at $R \Rightarrow y = 0 = \cancel{y_0} + v_{0y}\, t + \frac{1}{2} g t^2$

$$0 = v_0 \sin\theta_0\, t + \frac{1}{2} g t^2$$

$$v_0 \sin\theta_0 \cancel{t} = -\frac{1}{2} g t^{\cancel{2}}$$

$$t = -\frac{2 v_0 \sin\theta_0}{g}$$

(Remember g is $-9.8\,ms^{-2}$)

$$X = R = \cancel{X_0} + v_{0x} t + \cancel{\frac{1}{2} g_x t^2}$$

$$R = v_0 \cos\theta_0 \left(\frac{2 v_0 \sin\theta_0}{-g} \right)$$

$$R = \frac{2 v_0^2 \sin\theta_0 \cos\theta_0}{-g}$$

a long forgotten trig identity: $2\sin\theta_0 \cos\theta_0 = \sin 2\theta_0$

$$R = \frac{v_0^2 \sin 2\theta_0}{-g}$$

NB for maximum range we want

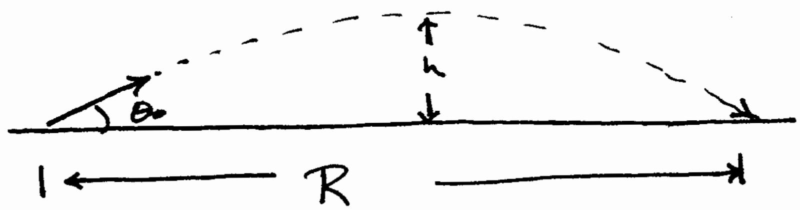
$\frac{v_0^2 \sin 2\theta_0}{g}$ to be maximum

For a given v_0, this will be the case when

$\sin 2\theta_0 = 1$ (biggest value sin can have

$2\theta_0 = 90°$

$\theta_0 = 45°$ not v. surprising



What is max. height; at $y = h$ $v_y = 0$

$$v_y = v_{0y} + g t$$

$$0 = v_0 \sin\theta + g t$$

$$t = \frac{v_0 \sin\theta_0}{-g}$$

$$y = h = \cancel{y_0}^{\,0} + v_{0y} t + \frac{1}{2} g t^2$$

$$= v_0 \sin\theta_0\, t + \frac{1}{2} g t^2$$

Subst for t

$$h = v_0 \sin\theta_0 \left(\frac{v_0 \sin\theta_0}{-g}\right) + \frac{g}{2}\left(\frac{v_0 \sin\theta_0}{-g}\right)^2$$

$$= \frac{v_0^2 \sin^2\theta_0}{-g} + \frac{\cancel{g}\, v_0^2 \sin^2\theta_0}{-2g^{\cancel{2}}}$$

$$= \frac{v_0^2 \sin^2\theta_0}{-2g}$$

A fly ball is hit to the outfielder 300 ft from home plate w/ runner on 3rd base. Outfielder is capable of throwing ball with $v_0 = 90\,\text{mph} = 132\,\text{ft s}^{-1}$ at what angle should he throw to home:

$$2\theta_0 = \sin^{-1}\frac{Rg}{v_0^2}$$

nb $g = 32\,\text{ft s}^{-2}$

$$\theta_0 = \frac{1}{2}\sin^{-1}\frac{Rg}{v_0^2}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{300\,\text{ft}\cdot 32\,\text{ft s}^{-2}}{[132\,\text{ft s}^{-1}]^2}\right)$$

$$\frac{1}{2}\sin^{-1} 0.55$$

Now $\sin\theta = 0.55$ at 33° and 147°

so $\theta_0 = 16.5^\circ$ or 73.5°

Outfielder can take his pick but

$$t = \frac{2 v_0 \sin\theta_0}{g}$$

$\theta_0 = 16.5^\circ$

$$t = \frac{2\cdot 132\,\text{ft s}^{-1}\cdot 0.28}{32\,\text{ft s}^{-2}}$$

$t = 2.3\,\text{s}$

$$h = \frac{(132\,\text{ft s}^{-1})^2 \sin^2 16.5^\circ}{32\,\text{ft s}^{-2}}$$

$h = 44\,\text{ft}$

$\theta_0 = 73.5^\circ$

$$t = \frac{2\cdot 132\,\text{ft s}^{-1}\cdot 0.95}{32\,\text{ft s}^{-2}}$$

$t = 7.9\,\text{s}$

$$h = \frac{(132\,\text{ft s}^{-1})^2 \sin^2 73.5^\circ}{32\,\text{ft s}^{-2}}$$

$h_2 = 500\,\text{ft}$

What if $h \neq 0$ initial   Same process, but slightly diff't answer

$$v_x = v_0 \cos\theta_0 \qquad v_y = v_0 \sin\theta_0 - g t$$

$$x = v_0 \cos\theta_0 t \qquad y = y_0 + v_0 \sin\theta_0 t - \tfrac{1}{2} g t^2$$

Range $\Rightarrow y = 0$

$$0 = y_0 + v_0 \sin\theta_0 t - \tfrac{1}{2} g t^2$$

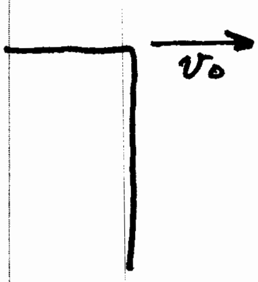
Now

$$t = \frac{-v_0 \sin\theta_0 \pm \sqrt{v_0^2 \sin\theta_0^2 + 2 g y_0}}{-g}$$

$$t = \frac{v_0 \sin\theta_0 \pm \sqrt{v_0^2 \sin\theta_0^2 + 2 g y_0}}{g}$$

$$x = R = v_0 \cos\theta_0 t = \frac{v_0^2 \cos\theta_0 \sin\theta_0 \pm v_0 \cos\theta_0 \sqrt{\ }}{g}$$

at $\theta_0 = 0$ $\cos\theta_0 = 1$ $\sin\theta_0 = 0$



$$R = \frac{v_0 \cos\theta_0 \sqrt{2 g y_0}}{g}$$

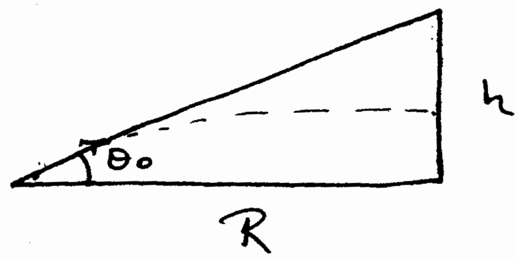
$$R = v_0 \sqrt{\frac{2 y_0}{g}}$$

Remember watermelon queen $v_0 = 5\,ms^{-1}$ $y_0 = 15\,m$

$$R = 5\,ms^{-1} \sqrt{\frac{2 \cdot 15\,m}{9.8\,ms^{-2}}} = 8.75\,m$$

(~ 30 ft)

MONKEY
DEMO



$$\tan\theta_0 = \frac{h}{R}$$

$$\cos\theta_0 = \frac{R}{(h^2+R^2)^{1/2}}$$

$t = 0$

BULLET

$x_0^b = 0 \qquad y_0^b = 0$

$v_{0x}^b = v_0 \cos\theta_0$

$v_{0y} = v_0 \sin\theta_0$

MONKEY

$x_0^m = R \qquad y_0^m = h$

$v_{0x}^m = v_{0y}^m = 0$

(Only need to worry abt y line)

$t = t_f$

$x^b = R \qquad y^b = ?$

$y^b = \cancel{y_0^b}^{\,0} + v_{0y}^b t + \frac{1}{2} g t^2$

$y^b = v_0 \sin\theta_0 t + \frac{1}{2} g t^2$

$x^b = \cancel{x_0^b}^{\,0} + v_{0x} t + \cancel{\frac{1}{2} a_x t^2}^{\,0}$

$x^b = R = v_0 \cos\theta_0 t$

$t = \frac{R}{v_0 \cos\theta_0}$

$x^m = R \qquad y^m = ?$

$y^m = y_0^m + \cancel{v_{0y}^m t} + \frac{1}{2} g t^2$

$y^m = h + \frac{1}{2} g t^2$

Subst for t in y^b eqn $\qquad y^m$ eqn :

$y^b = \cancel{v_0} \sin\theta_0 \left(\frac{R}{\cancel{v_0}\cos\theta_0}\right) + \frac{g}{2}\left(\frac{R}{v_0\cos\theta_0}\right)^2 \qquad y^m = h + \frac{g}{2}\left(\quad\right)^2$

$= R\tan\theta_0 + \frac{gR^2}{2v_0^2\cos^2\theta_0}$

$y^b = \cancel{R}\frac{h}{\cancel{R}} + \frac{gR^2}{2v_0^2\cos^2\theta_0} \iff y^m = h + \frac{gR^2}{2v_0^2\cos^2\theta_0}$