BERNOULLI'S EQN (Johann Bernoulli 1667-1748
Daniel Bernoulli 1707-17[illegible]

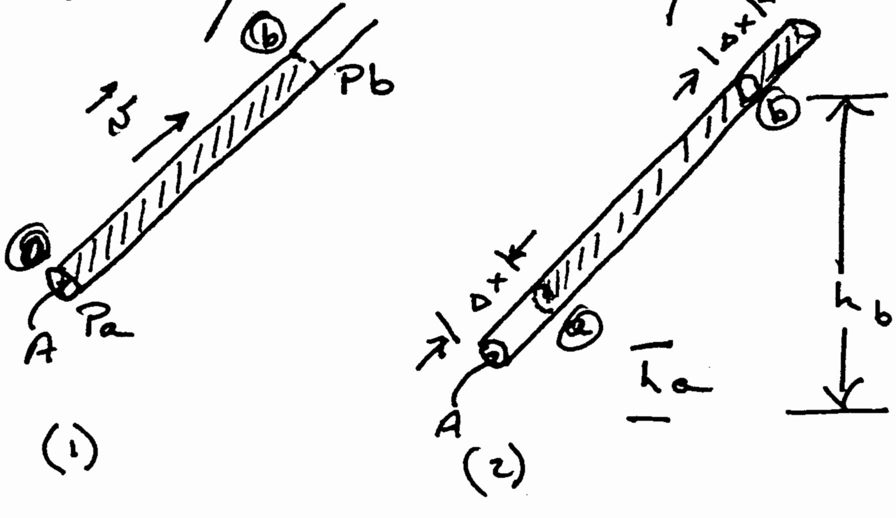
1) Incompressible fluid

2) Nonviscous fluid (no dissipation of energy in flow - no "internal friction")

3) Streamline flow

4) Steady-state flow (v at any given point does not change)

Under these assumptions consider a section of pipe, inclined wrt gravity, having constant cross-sectional area A + flow velocity v



Net force on the section of fluid will be

$$\text{net } F = F_a - F_b = P_a A - P_b A$$
$$= (P_a - P_b) A$$

If between (1) + (2) the fluid section moves distance Δx then the work done

$$W = \int \vec{F} \cdot \vec{ds} \quad \text{(remember?!)}$$

or in the simple situation where F = const and in direction of motion

$$W = F_{net}\, \Delta x$$

Substituting for the force

$$W = (P_a - P_b) \cdot A \Delta x$$

but $A\Delta x$ is the volume ΔV of fluid leaving the section of pipe so

$$W = (P_a - P_b)\, \Delta V$$

Now reach back again into the dusty Fall qtr. corners of your brain and recall the relation between work and potential energy. Work done = increase in potential energy ΔU.

$$\Delta U = mg(h_2 - h_1)$$

and

$$m = \rho \Delta V$$

thus

$$(P_a - P_b)\cancel{\Delta V} = W = \Delta U = \rho \cancel{\Delta V}(h_b - h_a)$$

or rearranging

$$P_a + \rho g h_a = P_b + \rho g h_b$$

In the case of <u>constant</u> <u>velocity</u>:

$$\underset{\text{Pressure}}{P} + \underset{\substack{\text{Potential energy} \\ \text{per unit volume}}}{\rho g h}$$ will be constant everywhere in the fluid.

What if $v \neq$ constant, recall KE

$$KE = \frac{1}{2} m v^2$$

If v changes, kinetic energy of the fluid changes; this is easily incorporated into the above

$$KE/\Delta V = \frac{1}{2}\rho v^2$$

and we have <u>BERNOULLI'S</u> <u>EQN</u>

$$P_a + \underbrace{\rho g h_a + \frac{1}{2}\rho v_a^2}_{\substack{\text{This is total} \\ \text{mechanical energy} \\ \text{per unit volume} \\ \frac{KE+PE}{\Delta V}}} = P_b + \rho g h_b + \frac{1}{2}\rho v_b^2$$

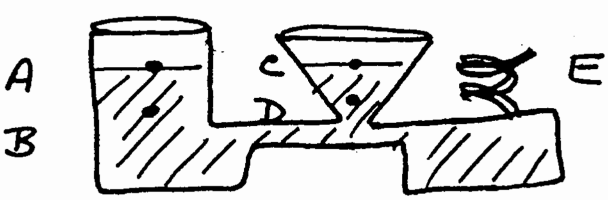
Consequences of Bernoulli's Eqn

1) Static Case ($v = 0$)

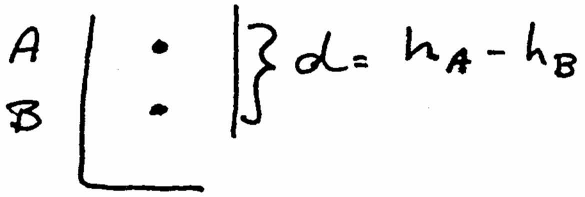
<u>Bernoulli</u>:

$$P + \rho g h = \text{constant}$$

Consider some fancy fluid container



Look first at the cylinder. According to Bernoulli's eqn



$$P_A + \rho g h_A = P_B + \rho g h_B$$

$$P_A = P_{atm}$$

$$P_B = P_{atm} + \rho g (h_A - h_B)$$

or substituting as above

$$P_B = \underbrace{P_{atm}}_{\text{atmospheric pressure}} + \underbrace{\rho g d}_{\text{Potential Energy difference}}$$

<u>e.g.</u> In one section of <u>20,000 Leagues Under the Sea</u> the Nautilus reaches a depth of 8000 fathoms (1 fathom = 6') ≈ 14,630 m

$$P_{NAUTILUS} = P_{atm} + \rho g d$$

Recall $\rho_{seawater} = 1025\ kg/m^3$

$$P_{NAUTILUS} = 1.013\times10^5\,Pa + 1025\ kg/m^3 \cdot 9.8\,m\,s^{-2} \cdot 14{,}630\,m$$

$$= 1.013\times10^5\,Pa + 1.47\times10^8\ kg/m\,s^2$$

$$= 1.47\times10^8\ Pa$$

$$\approx 1450 \text{ atmospheres!}$$

One square meter would feel the "weight" of 147 million newtons or 33 million pound

Two other inferences from Bernoulli's eqn. Consider second column shaped like a funnel: What is pressure at point $\underline{D}$ (if $h_D = h_B$)

Plug into Bernoulli's Eqn:

$$P_D = P_{atm} + \rho g d = P_B$$

$\Rightarrow$ The pressure at the same depth at two places in a static fluid will always be the same.

Similarly look at levels of fluid in each part of the vessel. Since $P_A = P_C = P_E = P_{atm}$ the level will be the same regardless of other circumstances n.b. Error in text 13.11 $h_B \to h_H$

Acceleration has a similar effect - effective weight

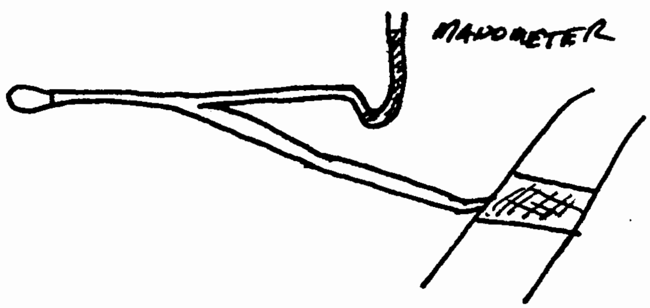
$$mg \to m(g+a)$$

$$P_{Brain} = P_{heart} + \rho(g+a)(h_{heart} - h_{Brain})$$

$a = 2-3 \times g \Rightarrow$ arteries in brain collapse leading to unconsciousness

This is the reason that you feel light headed when you stand up suddenly.

<u>SPHYGMOMANOMETER</u>



The manometer (now usually mechanical dial pressure gauge but sometimes you still see mercury column)

Pump cuff up until flow of blood is cut off

<u>SYSTOLIC</u> - peak pressure is measured as artery just opens. Artery is restricted $\Rightarrow v$ is high (eqn of continuity) and flow is turbulent (noisy)

GRAVITY + CIRCULATION

When reclining the arterial pressure in a person's system is fairly uniform; a typical value is

$$P = 13{,}3\ kPa = 13{,}300\ Pa$$

There are small differences due to viscosity or frictional losses between heart + head or feet. BERNOULLI'S EQN

$$P_{Feet} = P_{Heart} + \rho g h_{Heart} = P_{Brain} + \rho g h_{Brain}$$

HA : $h_{Heart} = 1.37\ m \qquad h_{Brain} = 1.83\ m$

$$\rho = 1060\ kg\,m^{-3} \qquad P_{Heart} = 13.3\ kPa$$

$$P_{Feet} = 13.3\ kPa + 1060\ kg\,m^{-3} \cdot 9.8\ m\,s^{-2} \cdot 1.37\ m$$

$$13.3\ kPa + 14.2\ kPa = \underline{27.5\ kPa}$$

(Nearly twice as high!)

But

$$P_{Brain} = P_{heart} + \rho g (h_{Heart} - h_{Brain})$$

$$= 13.3\ kPa + 1060\ kg\,m^{-3} \cdot 9.8\ m\,s^{-2}\ (1.37 - 1.83)\ m$$

$$= 13.3\ kPa - 4.8\ kPa = \underline{\underline{8.5\ kPa}}$$

DYNAMICAL CONSEQUENCES OF BERNOULLI'S EQN

Recall Bernoulli's Eqn - full blown treatment

$$P + \rho g h + \frac{1}{2}\rho v^2 = \underline{\text{constant in a streamline}}$$

if we consider effects at the same height then there is a direct relation between Pressure + velocity

$$P + \frac{1}{2}\rho v^2 = \text{const}$$

This says that the pressure must drop if the velocity increases in a fluid. This is a direct consequence of conservation of energy.

Increase in KE $\Rightarrow$ work must be done

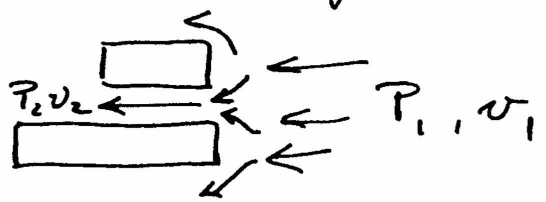
$\Downarrow$

$F_{NET} \rightarrow$

$P_1\ v_1$ $\qquad$ $P_2 < P_1$, $v_2 > v_1$

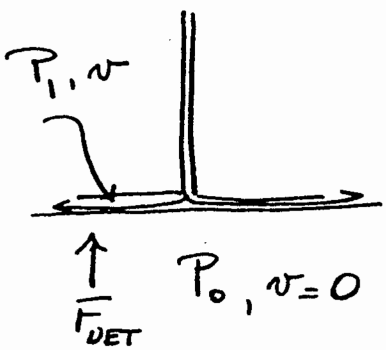
There must be a <u>net force</u> on the fluid. This force is provided by the fluid pressure. High net force means there must be a pressure decrease along the direction of motion

Truck passes you on the highway:



By Eqn of continuity $v_2 > v_1$ since area is restricted between car + truck. Bernoulli's eqn then requires $P_2 < P_1$. There will be pressure difference between inside + outside fluid, hence a net force <u>inward</u>.

<u>PLATE</u>



$$P_0 = P_1 + \frac{1}{2}\rho v^2$$

$$P_0 - P_1 = \frac{1}{2}\rho v^2$$ pressure diff

produces net $F = (P_0 - P_1) \cdot A_{DISK}$

What velocity is necessary to hold up the plate

Need $F = W_{PLATE} = m_{PLATE}\, g$

$$F = (P_0 - P_1) \cdot A_{disk} = \frac{1}{2}\rho v^2 A_{DISK} = m_{plate}\, g$$

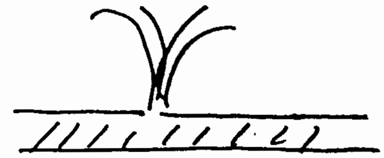
$$v^2 = \frac{2\, m_{plate}\, g}{\rho_{air} A_{DISK}}$$

$\rho_{air} = 1.3\ kg/m^3$ $m_{plate} = 0.7\ kg$ $A_{disk} = 0.02,$

$$v = 23\ m s^{-1}$$

Ex Pipe springs leak, spurts water into air $l = 3\,m$; what is Pressure P in pipe

Apply Bernoulli at hole



$$P + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_{atm} + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Look at flow right at hole $h_2 \approx h_1$, water in pipe static $\Rightarrow$ $v_1 = 0$

$$P = P_{atm} + \frac{1}{2}\rho v_2^2$$

Ohmigod! What do I do now? How do I get v_2? Some may recall $v^2 = v_0^2 + 2gh$ but better, recall cons of energy

at hole $\quad \frac{KE}{Vol} = \frac{1}{2}\rho \cdot v_2^2 \quad$ let $PE = 0$

at $h \quad \frac{PE}{Vol} = \rho g h \quad KE = 0$

$$\frac{1}{2}\cancel{\rho} v_2^2 = \cancel{\rho} g h \qquad v = \sqrt{2gh}$$

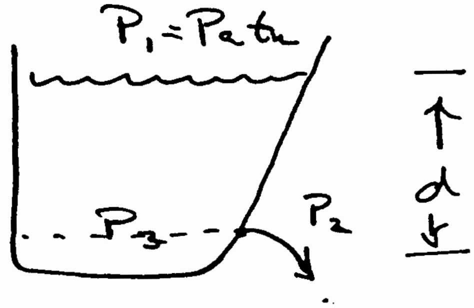
$$v_2 = \sqrt{2 \cdot 9.8 \cdot 3} = 7.67\ m\,s^{-1}$$

$$P = 1.013\times10^5\,Pa + \frac{1}{2}\cdot 1000\ kg\,m^{-3}\ 58.8\ \frac{m^2}{s}$$

$$P = 1.013\times10^5\,Pa + 0.294\times10^5\,Pa = 1.307\times10^5$$

Overpressure $P - P_{atm} = 0.29\times10^5\,Pa$
$= 0.29\ atm$

Bucket w/ hole



Bernoulli

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

$$= P_3 + \rho g h_3 + \frac{1}{2}\rho v_3^2$$

$v_1 = 0 \qquad P_1 = P_{atm} \quad P_2 = P_{atm}$

$v_3 = 0$

$$\overset{= P_{atm}}{\cancel{P_1}} + \rho g h_1 = \overset{= P_{atm}}{\cancel{P_2}} + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

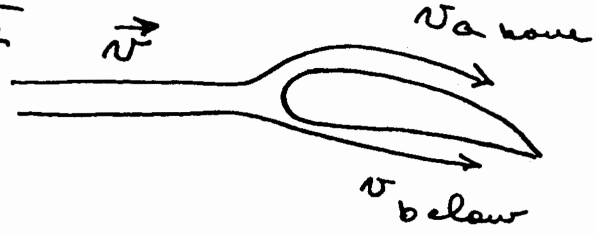
$$\frac{1}{2}\cancel{\rho} v_2^2 = \rho g (h_1 - h_2) = \cancel{\rho} g d$$

$$v^2 = 2gd \qquad v = \sqrt{2gd}$$

Could equally well work from

$$P_3 = P_{atm} + \rho g d$$

FLIGHT



α = angle of attack

v is the "air speed"

$$v_a > v_b$$

Bernoulli

$$P_a + \rho g h_a + \frac{1}{2}\rho v_a^2 = P_b + \rho g h_b + \frac{1}{2}\rho v_b^2$$

Thin wing $h_a \approx h_b$

$$P_b - P_a = \frac{1}{2}\rho\left(v_a^2 - v_b^2\right)$$

Since there is a net pressure difference ($P_a < P_b$) there is a net force - "LIFT"

$$F_L = (P_b - P_a)\cdot A_{WING} = A\frac{\rho}{2}\left(v_a^2 - v_b^2\right)$$

$v_a, v_b \propto v$ complicated physics but we can write

$$F_L = A\, C_L \frac{\rho}{2} v^2$$

C_L is "lift coeff't" – we hide all the physics here; can see C_L depends upon α, angle of attack, wing shape etc. Generally measure in wind tunnel.

$$F_L \propto A v^2$$

high v (jet) low A
low v (prop) high A

For flight to occur need

$$F_L \geq W \quad \text{where } W \text{ is weight of flier}$$

If all fliers had same shape

$$W \propto \text{size}^3 \quad \underset{l^3}{(\text{volume})}$$

$$A \propto \text{size}^2 \quad \underset{l^2}{(\text{area})}$$

Then

$$F_L = A C_L \rho v^2/2 \propto l^2 v^2$$

$$W \propto l^3$$

for $F_L = W$ we have

$$l^2 v^2 \propto l^3$$

or

$$v \propto l^{1/2}$$

Bigger birds need larger velocity for flight

$$v_{swift} \approx 6 \, m \, s^{-1} \quad (13\tfrac{1}{2} \text{ mph})$$

for takeoff $\quad l_{swift} \approx 7.5\,cm \quad (3'')$

A typical ostrich may be 6'

$$l_{ostrich} \approx 180\,cm$$

$$v_{ostrich} = \left(\frac{l_{ostrich}}{l_{swift}} \right)^{1/2} \cdot v_{swift}$$

$$\approx 5\, v_{swift}$$

$$= 30\, m\, s^{-1} \quad \text{nearly } 70 \text{ mph}$$

ostriches can run v. fast, up to 40 mph for short stretches alas!