Motion of Ideal Fluids

"Ideal" means:

1. Steady flow (no change with time, c.f. turbulent)
2. Incompressible (ρ constant)
3. Non-viscous flow (i.e. no fluid-fluid friction)

   - friction transforms kinetic energy $\rightarrow$ internal heat energy

4. Irrotational flow - no vortices

   e.g. matchstick in flow does not rotate

Fluid elements follow stream lines

Streamlines are $\parallel$ to velocity, do not intersect

$\therefore$ Streamlines on boundary of area A_1 define "flow tube" to A_2

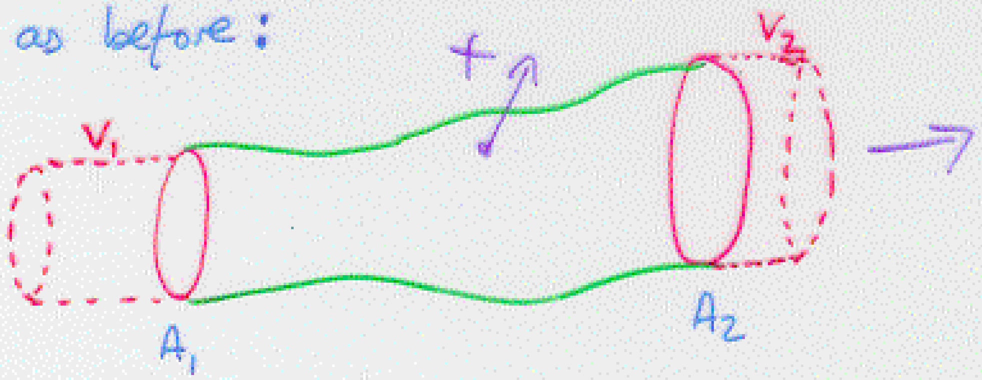
(no fluid elements cross boundary of tube)

As flow spreads out, fluid slows down.

Equation of Continuity (Mass Conservation)

Take flow tube in ideal flow

e.g. as before:



In 1s, volume entering through A_1 is $A_1 v_1$

For steady, incompressible flow, this

= volume exiting A_2, i.e. $A_2 v_2$

$\Rightarrow A_1 v_1 = A_2 v_2 = R$ (volume flow rate in m^3/s)

Mass flow rate $= \rho \times$ volume $= \rho R$ (kg/s)

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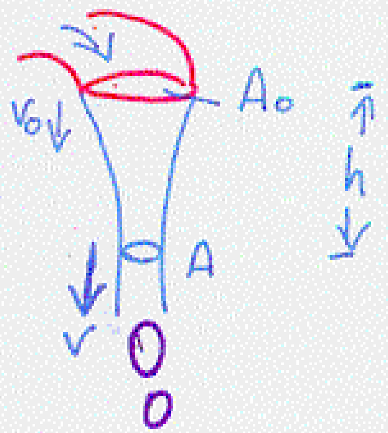
e.g. "Necking" of water falling from faucet



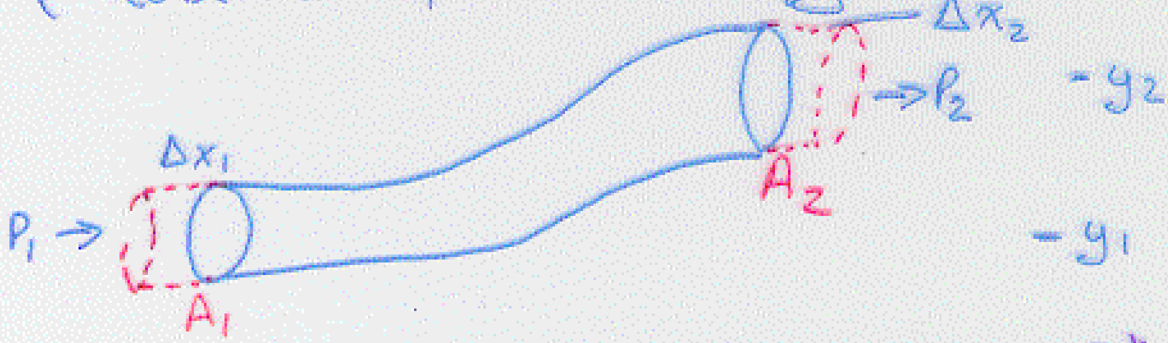
Free fall speed $v^2 = v_0^2 + 2gh$

Area at h: $A = \frac{A_0 v_0}{v} = R/v$

$$\Rightarrow A = \frac{R}{\sqrt{v_0^2 + 2gh}}$$

Bernoulli's Equation for Ideal Flows

(Conservation of Work - energy)



In time Δt, mass m of fluid travels $\Delta x_1 = v_1 \Delta t$

Work done to push m into tube $= F\Delta x_1 = P_1 A_1 \Delta x_1$

Work done by mass leaving tube $= F\Delta x_2 = P_2 A_2 \Delta x_2$

But $A_1 \Delta x_1 = A_2 \Delta x_2 =$ volume ΔV

Initial k.e. $= \frac{1}{2} m v_1^2 = \frac{1}{2} \rho \Delta V v_1^2$

Initial p.e. $= mgy_1 = \rho \Delta V g y_1$

Net work $(P_2 - P_1)\Delta V =$ change in (k.e. + p.e.)

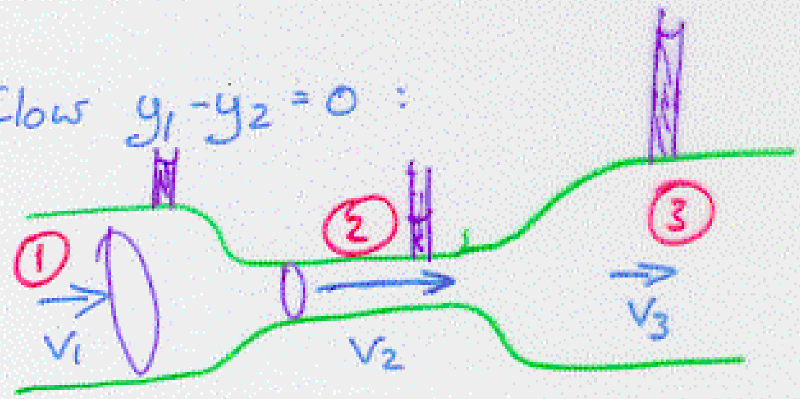
i.e. $(P_2 - P_1)\Delta V = (\frac{1}{2}\rho v_1^2 + \rho g y_1)\Delta V - (\frac{1}{2}\rho v_2^2 + \rho g y_2)$

$$\Rightarrow \boxed{P + \tfrac{1}{2}\rho v^2 + \rho g y = \text{constant} \quad \text{Bernoulli's Equation}}$$

Consequences of Bernoulli's Eqn.

1. Static case : $v_1, v_2 = 0 \Rightarrow P_1 - P_2 = \rho g (y_2 - y_1)$
   as before

2. Level flow $y_1 - y_2 = 0$:



From continuity eqn. $v_2 > v_1$ as flow narrows

$\therefore \quad P_2 = P_1 - \frac{1}{2}\rho (v_2^2 - v_1^2)$: pressure drops

Physically: Higher pressure P_1 does work to accelerate fluid from v_1 to v_2

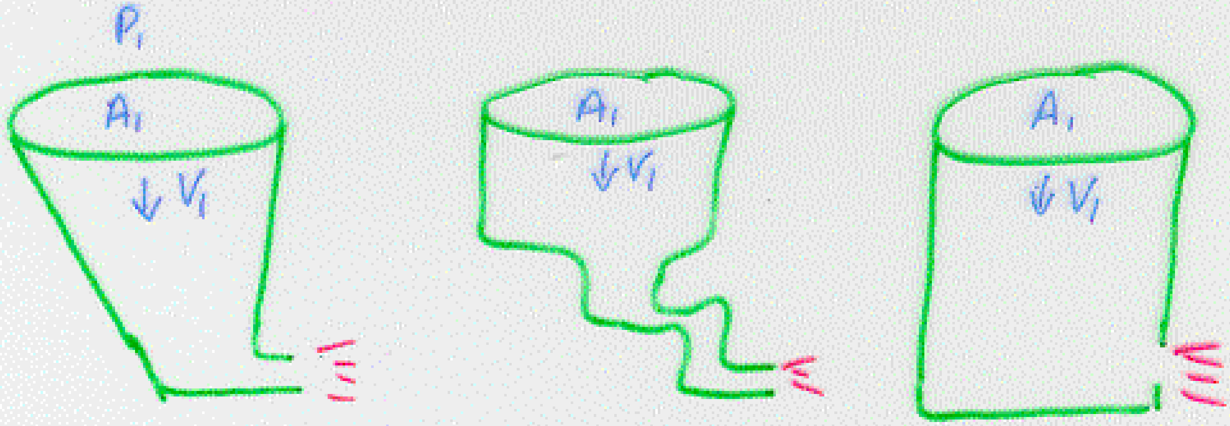
Then fluid in region 2 pushes on region 3 as it decelerates $\rightarrow$ higher pressure in region 3

For areas A_1 , A_2.

Continuity $\Rightarrow \quad v_2 = \frac{A_1 v_1}{A_2}$

$\Rightarrow$ Pressure drop $\Delta P = \frac{1}{2}\rho (v_1^2 - v_2^2) = \frac{1}{2}\rho v_1^2 \left(1 - \frac{A_1^2}{A_2^2}\right)$

3. General Cases



These flows are identical if A_1, A_2, P_1, h the same

Problems often give initial conditions
then ask for v_1, V_2, or flow rate

Use : $P_1 + \frac{1}{2}\rho_1 V_1^2 + \rho g h = P_2 + \frac{1}{2}\rho_2 V_2^2 + \rho g h_2$ (1)

To get V_1, V_2 must also use continuity

i.e. $A_1 V_1 = A_2 V_2 = R$ (2)

Can then eliminate V_2 and solve (1).

Note: Torricelli's experiment, $V_1 = \frac{A_2 V_2}{A_1} << V_2$
so can neglect V_1 for easy solution