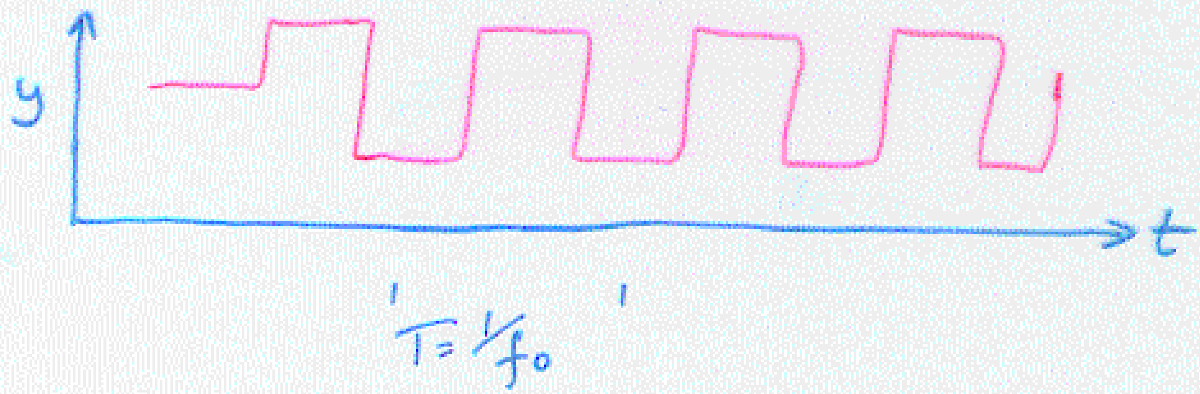
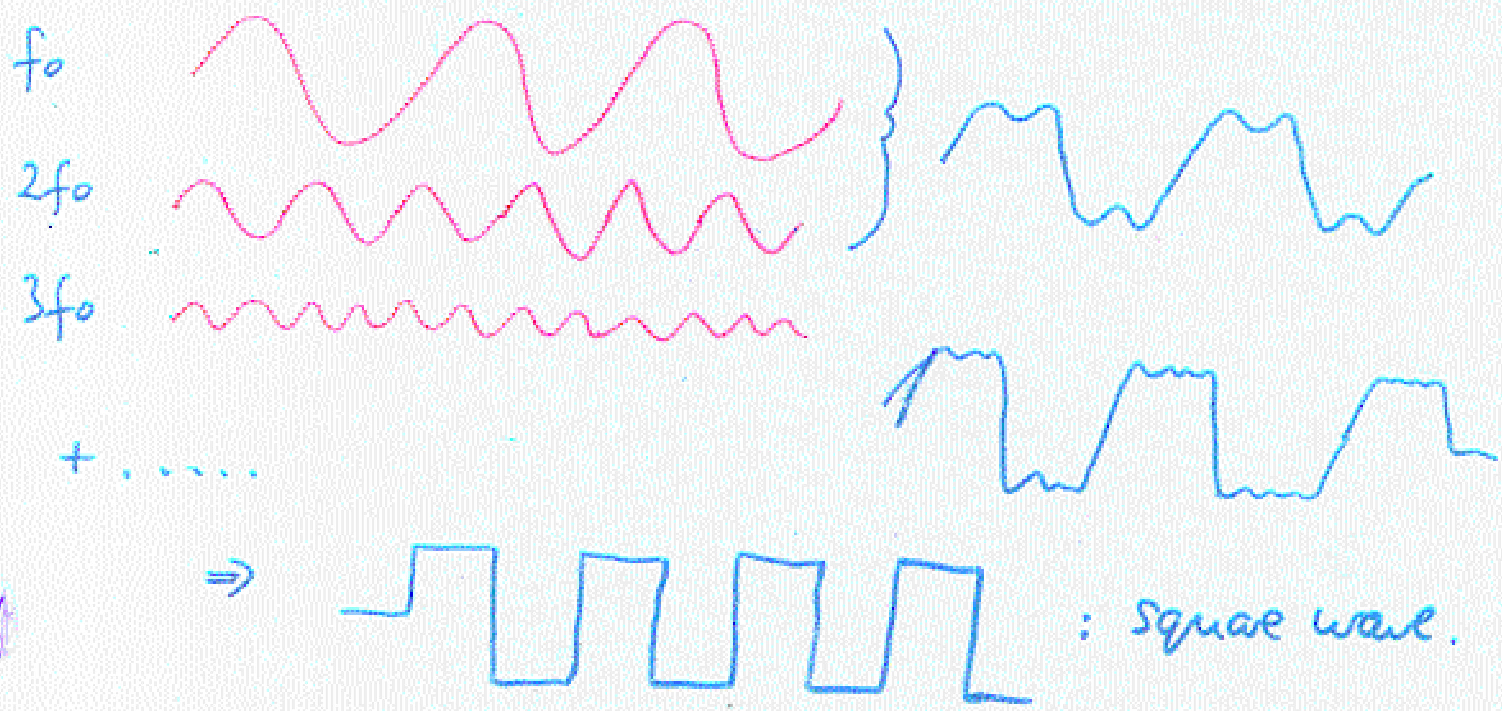
Superposition: Fourier Analysis.

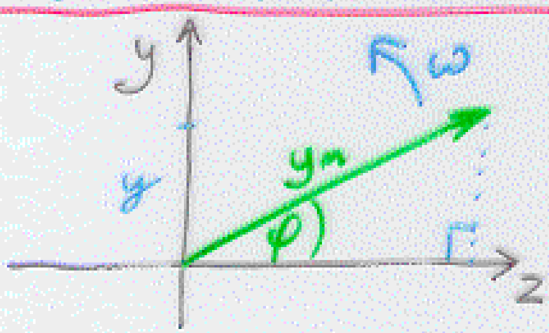
Every periodic wave has fundamental freq. f_0



Fourier (1807): Any periodic waveform can be made from superposition of harmonic waves with freqs. $f_0, 2f_0, 3f_0, 4f_0, \ldots$ and higher:



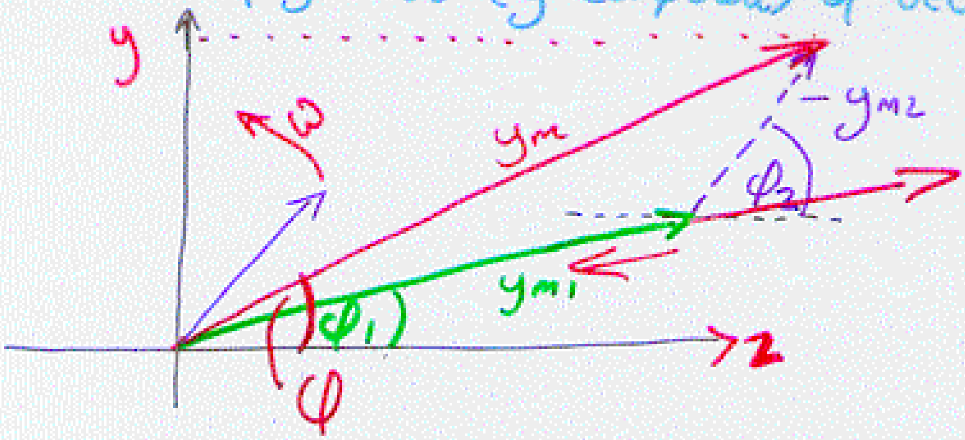
Phasors: Vector Addition of Harmonic Waves



At any point (take $x=0$): displacement y represented by projection ("shadow") of rotating vector, length y_m (ang. speed = ω)

i.e. $y = y_m \sin(\omega t + \phi)$

So, to add harmonic waves: first add the vectors, then take projection (y-component) of vector sum:



i.e. find y_m and ϕ, then $y = y_m \sin\phi$ (at $t=0$)

If $\omega_1 = \omega_2$ and wave direction same, then angles $(\phi_1 + \omega t)$, $(\phi_2 + \omega_2 t)$ keep constant phase diff. $(\phi_1 - \phi_2)$

$\Rightarrow$ resulting vector also rotates uniformly with $\omega = \omega_1 = \omega_2$.

$\Rightarrow$ Add 2 harmonic waves of same frequency results in another harmonic wave.

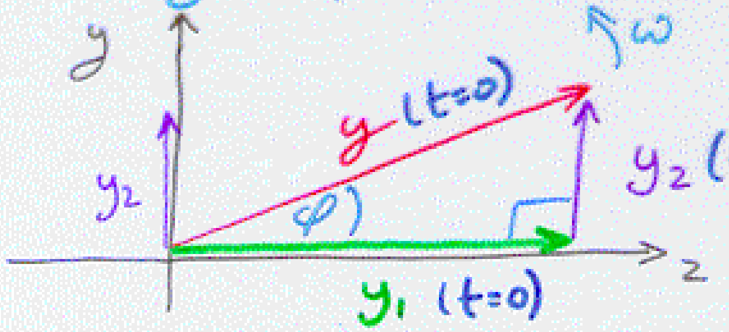
Example: Add two harmonic waves

$y_1 = 12 \sin(kx - \omega t) \quad (\phi_1 = 0)$

$y_2 = 5 \sin(kx - \omega t + \frac{\pi}{2})$

– what is $y = y_1 + y_2$?

Phasor diagram for $x = 0$ (draw at $t=0$):



Use easy trigonometry to find amplitude y_m, phase ϕ

Pythagoras (since $\phi_2 - \phi_1 = 90°$): $y_m^2 = 12^2 + 5^2$

$\Rightarrow \underline{y_m = 13}$

Phase angle ϕ given by $\tan\phi = \frac{y_2(0)}{y_1(0)} = \frac{5}{12}$

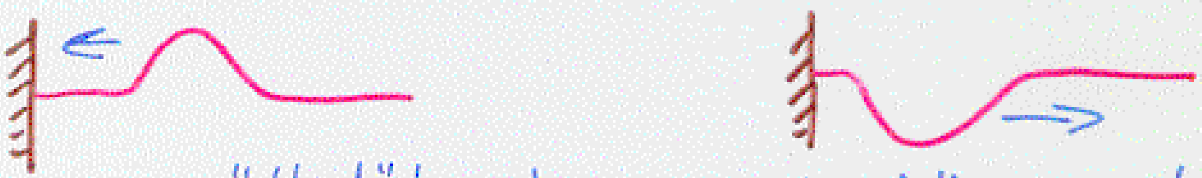
$\Rightarrow \phi = \underline{22.6° \times \frac{\pi}{180°}} = \underline{0.394 \text{ rad}}$

$\therefore$ resulting $y = y_m \sin(kx - \omega t + \phi)$!

$\underline{= 13 \sin(kx - \omega t + 0.394)}$

Also works for >2 vectors, e.g. 3-phase power

Reflection at a Boundary (Fig 17.16)



"Hard" boundary – node at the support,
reflected wave inverted

"Soft" boundary – antinode at the boundary
reflected wave continues unchanged

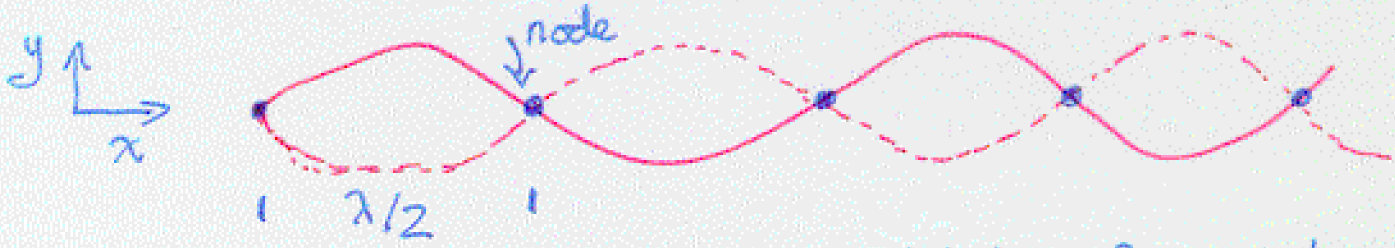
Standing Waves

Can superpose 2 equal waves traveling in opposite directions:

$$y_1 = y_m \sin(kx - \omega t) \; ; \; y_2 = y_m \sin(kx + \omega t)$$

$$\Rightarrow y = y_1 + y_2 = \overbrace{y_m \cdot 2 \sin kx}\, \cos \omega t$$

i.e. not a traveling wave!



Oscillation fixed in space with amplitude $= 2 y_m \sin kx$

Nodes where amplitude $= 0$ i.e. $\sin kx = \sin \frac{2\pi x}{\lambda} = 0$

when $kx = \frac{2\pi}{\lambda} = \pi, 2\pi, 3\pi, \ldots n\pi$

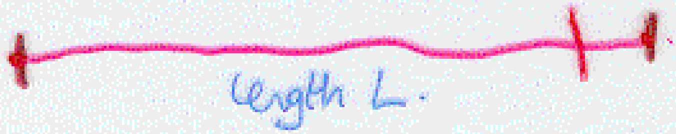
$\Rightarrow$ nodes at $x = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots \frac{n\lambda}{2}$

Antinodes where $\sin kx = \pm 1$ (displacement maximum)

$\Rightarrow kx = \frac{2\pi x}{\lambda} = \frac{1}{2}\pi, \frac{3\pi}{2}, \ldots (n + \frac{1}{2})\pi$

Standing Waves and Resonance

e.g. string fixed at both ends



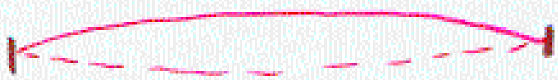
Wave reflects off each end and interferes to produce a standing wave with nodes at $x = \frac{n\lambda}{2}$.

If we choose frequencies such that ends of string naturally occur at nodes $\rightarrow$ RESONANCE

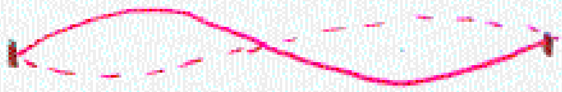
- stable pattern with energy $\propto (\text{amplitude})^2$

$\therefore$ Boundary condition for resonance : $L = \frac{n\lambda}{2}$

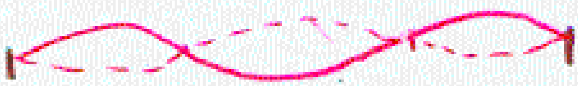
OR $\lambda = \frac{2L}{n}$ $\Rightarrow$ hear resonant frequencies $f = \frac{v}{\lambda} = \frac{vn}{2L}$



fundamental $n=1$



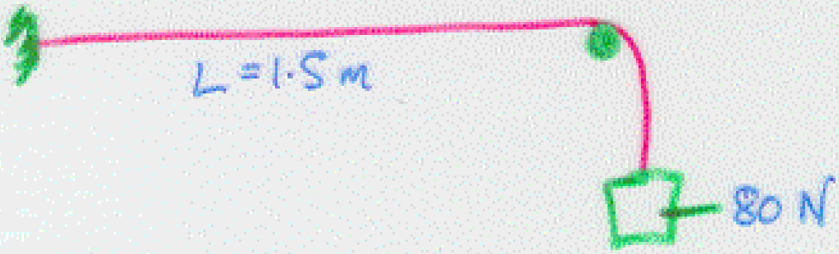
1st harmonic $n=2$



2nd harmonic $n=3$.

Example:

String stretched over $L = 1.5\,m$ with a tension $\tau = 80N$. What are the resonant frequencies (fundamental + harmonics) if $\mu = 2g/m$?



1. Resonant wavelengths have $L = \frac{n\lambda}{2}$ so $\lambda = \frac{2L}{n} = \frac{3.0}{n}$ m

   – does not depend on tension or mass of string!

2. Wave speed: $v^2 = \frac{\tau}{\mu} = \frac{80N}{2\times10^{-3}\,kg/m} \Rightarrow v = 200\,m/s$ (sound in air: 330m/s)

3. Resonance freq. $f = \frac{v}{\lambda} = \frac{v \cdot n}{2L} = \frac{200n}{3} = 66.7n$ Hz

For any point on string:

$$y = (2y_m) \sin kx \cos \omega t \quad \text{with } k = \frac{2\pi}{\lambda},\ \omega = 2\pi f$$

$$= (2y_m) \sin \frac{\pi n x}{L} \cos \frac{\pi v n t}{L}$$