KE and P.E. in SHM

$$x = A\cos(\omega t + \phi) \; ; \quad \omega^2 = k/m$$

$$\text{P.E. of spring} = \int F.dx = \tfrac{1}{2} k x^2 = \tfrac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

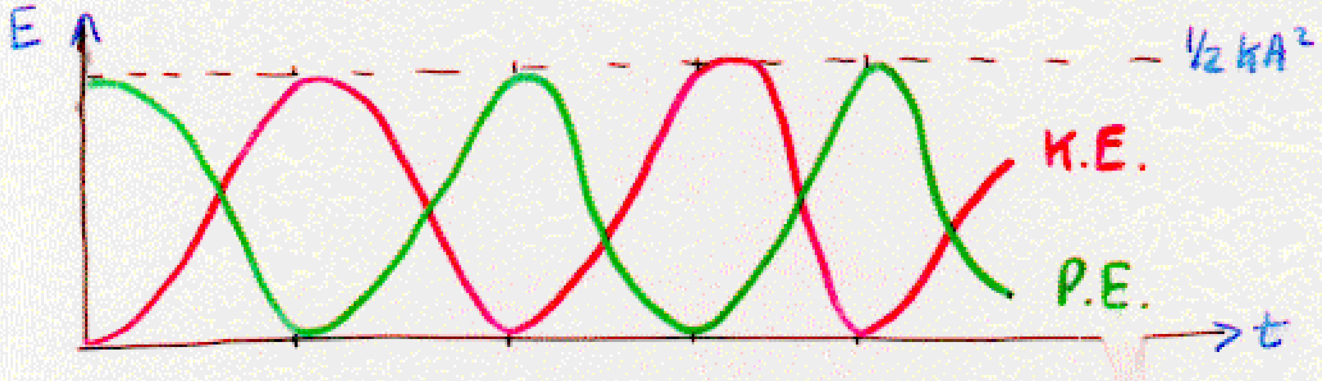
$$\text{K.E. of mass} = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \tfrac{1}{2} m \underline{\omega^2} A^2 \sin^2(\omega t + \phi)$$

But we know $\omega^2 = k/m$ from above

$$\Rightarrow KE = \tfrac{1}{2} \cancel{m} . \frac{k}{\cancel{m}} A^2 \sin^2(\omega t + \phi)$$

Total energy $\quad KE + PE = \tfrac{1}{2} k A^2 \underbrace{\left[\sin^2(\cdots) + \cos^2(\cdots) \right]}_{=1}$

$= \tfrac{1}{2} k A^2$, constant !



At ends of motion, $v = 0$, $x = \pm A$ and $PE = \tfrac{1}{2} k A^2$, $KE = 0$

At $x = 0$, $PE = 0$ and $KE = \tfrac{1}{2} m v^2 = \tfrac{1}{2} k A^2$.

e.g. A mass of 0.4kg vibrates on a spring with $k = 100\ kg/s^2$ (N/m)

0.71 m

Mass is given an initial "kick" of 0.5 m/s.

$\Delta l = 0.92 - 0.71 = 0.21$ for 0.2 kg

Find the amplitude of the motion, and the eqn. of motion.

$$\Rightarrow k = \frac{mg}{\Delta l} = \frac{0.2 \times 9.81}{0.21\,m} = 9.34\ N/$$

$v = 0.5\,m/s$ at $t=0$

$x=0$

$\frac{1}{2} \times 0.4 \times (0.5\,m/s)^2$

1. Initial $KE = \frac{1}{2}mv^2 = 0.05\,J$. initial $PE = 0$

$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2}{9.34}} = 0.919\,s$

So max. PE (when $x = \pm A$ and $v = 0$) is $\frac{1}{2}kA^2 = 0.05\,J$

$\Rightarrow A = 0.0316\,m$.

2. $x = A\cos(\omega t + \varphi)$. $\omega = \sqrt{\frac{k}{m}} = 15.8\ rad/s$

Speed $v = -\omega A \sin(\omega t + \varphi) = 0.5\,m/s$ at $t = 0$

So at $t=0$:

$$0.5\,m/s = -\omega A \sin\varphi \quad (1)$$
$$x = 0 = A\cos\varphi \quad (2)$$

(1), (2) only satisfied if $\varphi = -\pi/2$ $(-90°)$

Then $\sin\varphi = -1$ and we can check $v(0) = \omega A = 15.8 \times 0.316 = 0.5\,m/s$ ✓✓

So motion is $x(t) = 0.316\,m\,\cos(15.8t - \pi/2)$

i.e. $x = 0.316\,m\,\sin 15.8t$

Pendulum cont/d.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad (\text{small } \theta)$$

$$\Rightarrow \quad \theta = A\cos(\omega t + \varphi) \text{ with}$$

$$\text{period } T = 2\pi\sqrt{\frac{L}{g}}$$

Note:

- T depends on L, g only : NOT mass (c.f. spring)
  so child, adult on same swing have same T
- T constant as amplitude decreases (so pendulum is a good clock)
- can measure L, T to estimate "g" on earth and other planets.

Example : For a clock with $T = 2\,s$ (so 1 "tick" = 1 s)

Pendulum length $L = \frac{T^2}{4\pi^2}\cdot g = \frac{4}{4\pi^2} \times 9.81 = 0.993\,m$. on earth.

Bio-mechanics :

Can model {leg + foot} as pendulum for humans, animals

$\Rightarrow$ "natural" period T of leg

So "gait" or # steps/s $\propto f = \frac{1}{T} \propto \sqrt{\frac{g}{L}}$ ~ 120 step/min for adults

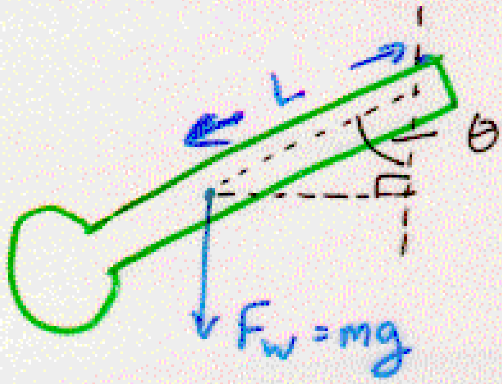
Compare: toddler, adult, giraffe on earth

astronaut, other life (?) on Mars, other planets.

Note Stride length $\propto L$ so walking speed v = stride × gait (~100 yd/min)

$$\propto L\sqrt{\frac{g}{L}} = \sqrt{gL}.$$

Solid Pendulums



e.g. baseball bat
golf club
leg of animal
(human, dinosaur....)

L = distance from pivot to center of mass

$$\text{Torque} = F_w . L \sin\theta = mgL\sin\theta$$

$$= \text{mom. of inertia} \times \text{ang. accel}$$

$$= \frac{I\,d^2\theta}{dt^2} \qquad \text{For small } \theta,\ \sin\theta \approx \theta$$

$$\Rightarrow I\frac{d^2\theta}{dt^2} = -mgL\theta$$

$$\text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{mgL}{I}\theta = -\omega^2\theta$$

(Note: for mass on string, $I = mL^2 \Rightarrow \omega^2 = g/L$ as before)

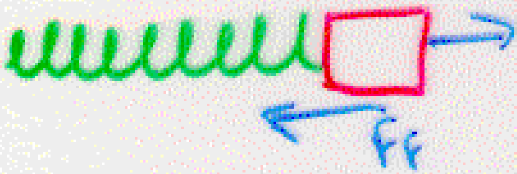
e.g. For rod length ℓ, dist. to c.m. $L = \ell/2$

$$\text{and } I = \int \rho A r^2 dr = \tfrac{1}{3} m\ell^2 \quad \text{(Table 8.3)}$$

$$\Rightarrow \omega^2 = \frac{mg\,\ell/2}{\tfrac{1}{3}m\ell^2} = \frac{3}{2}\frac{g}{\ell} \quad \left(\text{c.f. } \frac{g}{L} \text{ for mass on string}\right)$$

So for {leg + foot}, calculate dist. to c.m. L, moment of inertia $I = \sum mr^2$ } $\Rightarrow$ period T.

Damped Oscillations : Energy Loss



Mass on spring sliding on table

In reality, $E = PE + KE$ $\downarrow$ with time due to friction

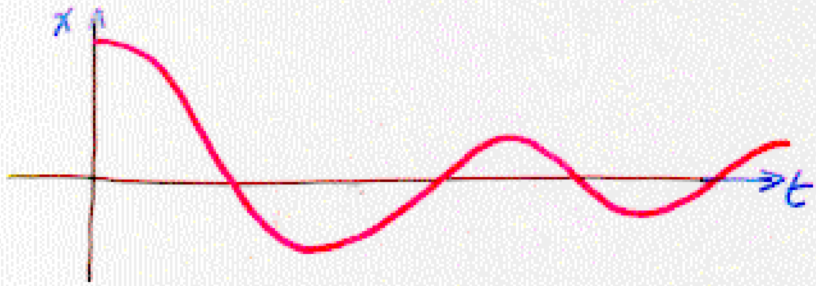
So since $E = \frac{1}{2} k A^2$, amplitude $A \downarrow$ also.

Energy lost = work done against friction

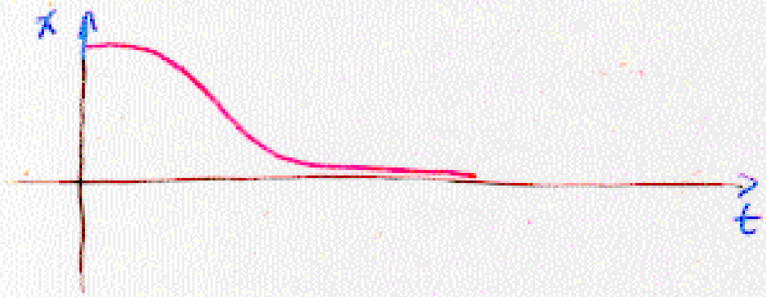
at a rate $\frac{dW}{dt} = F_F \cdot v$

$\Rightarrow$ new eqn. of motion $m\frac{d^2x}{dt^2} = -kx - F_F$

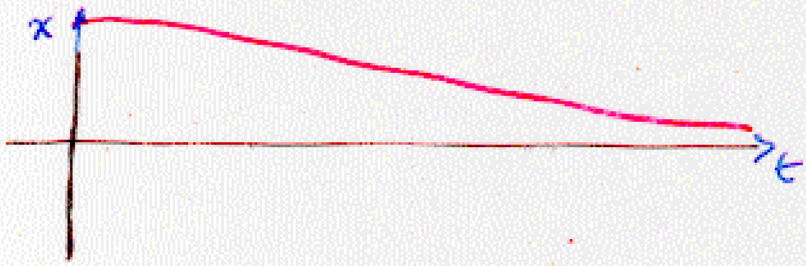
Solutions:



F_F small
"under-damped"



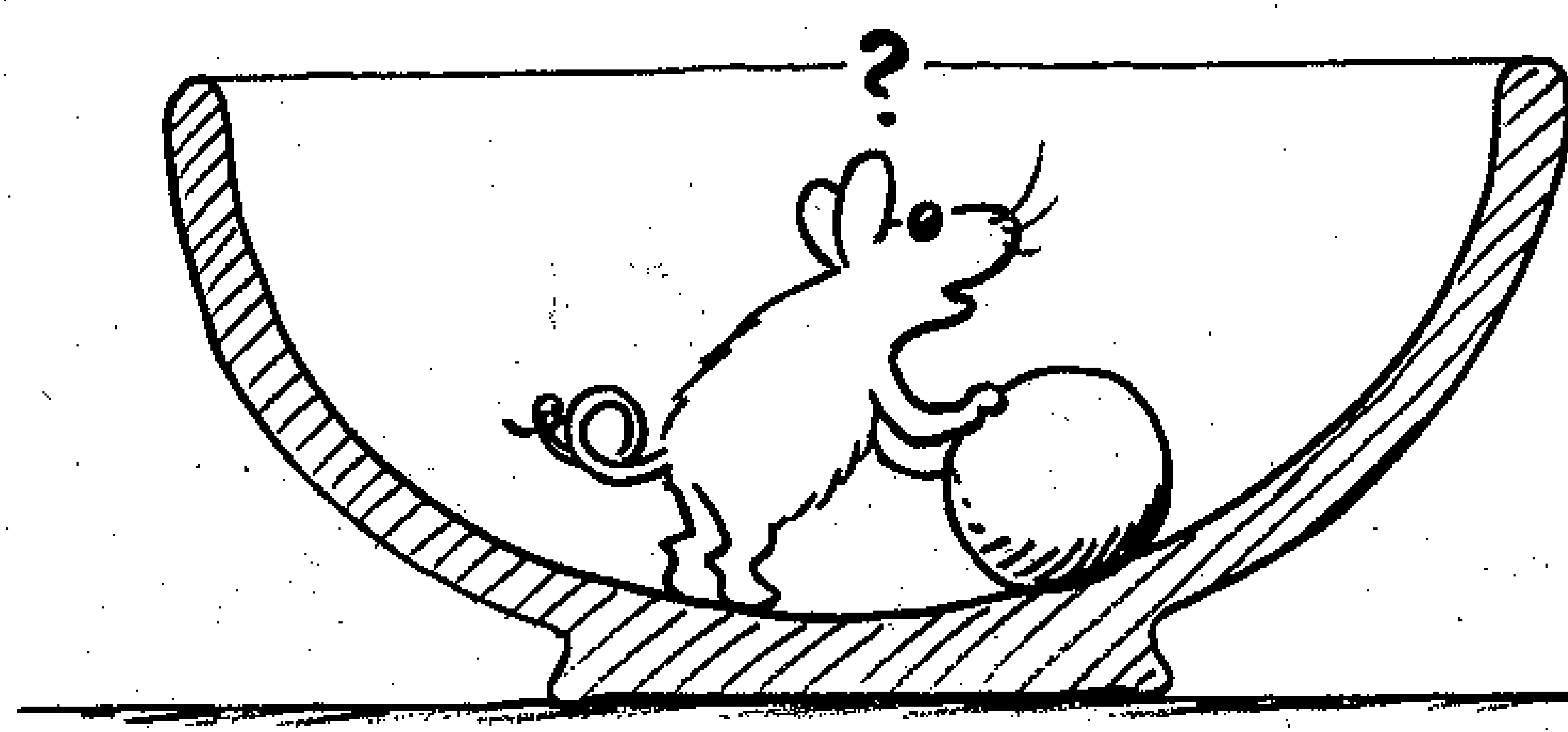
$F_F \approx kA$: "critically damped"
e.g. shock absorber



"over-damped"
F_F large

Meeky Mouse wants to get the ball bearing up and out of the bowl, but the ball is too heavy and the sides of the bowl too steep for Meeky Mouse to support the ball's weight. Using only its own strength without the help of levers and such, Meeky

a) can't get the ball bearing up and out
b) can get the ball bearing up and out (But how?)



Forced Oscillations and Resonance

Apply additional force(s) to oscillator

$$m\frac{d^2x}{dt^2} = -kx + F(t)$$

- If F constant (e.g. wind against tree, blow across tube)

  $\Rightarrow$ SHM as before with $\omega^2 = k/m$, freq $f = \omega/2\pi$.

- Often, $F(t)$ is also periodic with driving frequency f_0

  (e.g. road joints on freeway, push a swing, sing a note at wine glass) troops crossing bridge,

System absorbs energy from driving force when it is "in tune" with natural freq., i.e. $f_0 \approx f$

$\Rightarrow$ resonance : every cycle absorbs energy

and since $E = \frac{1}{2}kA^2$, amplitude $A \uparrow$ also until energy input = work done against friction.

(or something breaks!)