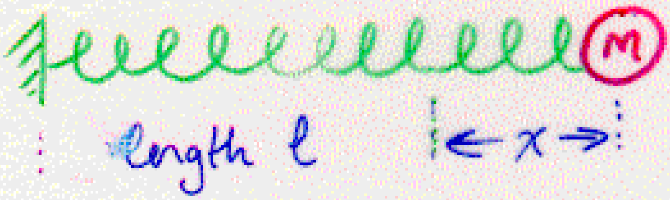
# Vibrations of mass on a spring (ch. 10)




When mass displaced by $x$, restoring force $F = -kx$
(Hooke)

From Newton II: $F = m\dfrac{d^2x}{dt^2} = -kx$

or accel. $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m} \cdot x$

(accel. $\propto$ $^{(-)}$displacement)

Solution: $x = A\cos(\omega t + \phi)$

Speed $v = \dfrac{dx}{dt} = -A\omega\sin(\omega t + \phi)$

Accel $a = \dfrac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x$

Solution works if $\omega^2 = \dfrac{k}{m}$ $[s^{-2}]$

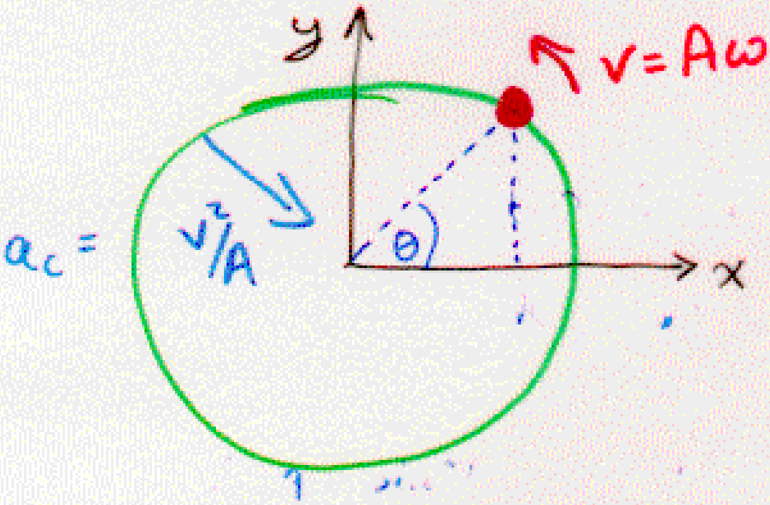
and constants $A$ [m], $\phi$ [rad] set by

initial conditions (e.g. at $t=0$:

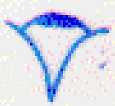
disp. $x = A\cos\phi$ (1)

speed $v = -A\omega\sin\phi$ ) (2)

# Relation to Circular Motion




Object moves in circle, radius $= A$ at uniform ang. speed $\omega$



$=$ View edge-on (e.g. Jupiter's moons)

At time $t$, angle $\theta = \omega t + \varphi$ ($= \varphi$ at $t=0$)

Observer sees only $x$-coordinate $x = A\cos\theta$

i.e. $x = A\cos(\omega t + \varphi)$

Also, $x$-component of centripetal accel $a_c = \frac{v^2}{A}$ is

$a_x = a_c \cos\theta = \frac{v^2}{A} \cdot \frac{x}{A} = \frac{v^2}{A^2} \cdot x = (-)\omega^2 x$ as before

Period of cycle $T = \frac{2\pi}{\omega}$. Frequency $f = 1/T = \frac{\omega}{2\pi}$ ($s^{-1}$ or Hertz).

Max. displacement along $x$-axis is $x = \pm A$ [m]
(when $\cos(\omega t + \varphi) = \pm 1$)

$A$ : amplitude of oscillation

$\varphi$ : Initial phase angle [rad] at $t=0$.

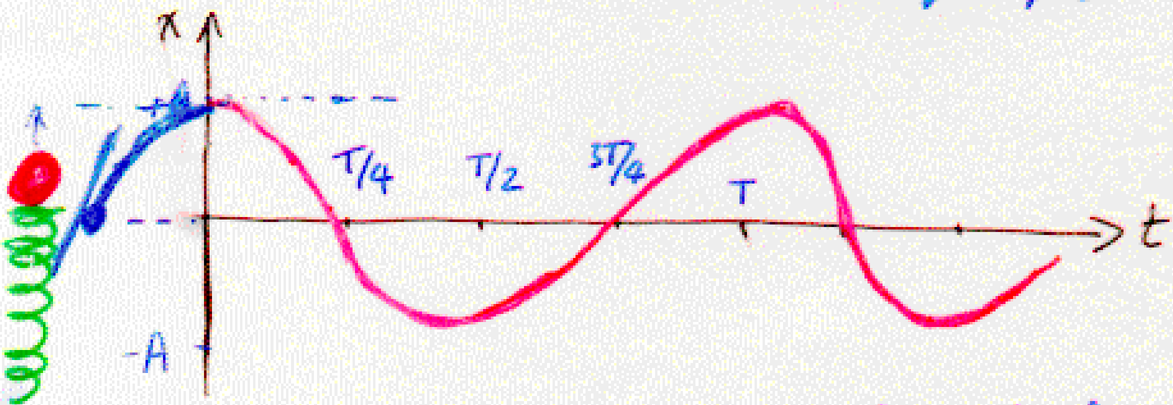
# Simple Harmonic Motion (SHM)

Many systems have a restoring force $\propto$ (−) displacement

e.g. child on swing, swaying trees, diatomic molecules

i.e. $F = m\frac{d^2x}{dt^2} = -kx$ or $\frac{d^2x}{dt^2} = -\omega^2 x$

$\Rightarrow$ Solution $x = A\cos(\omega t + \phi)$ : harmonic vibration about eqm. position ($x=0$) .....




... with ang freq. $\omega = \frac{2\pi}{T} = 2\pi f$ [rad/s]

and initial phase angle $\phi$.

If $\phi = 0$ (shown here), $x = +A$ at $t=0$

also $v = \frac{dx}{dt} = 0$ at $t=0$

(e.g. displace object to $x=A$, then let go)

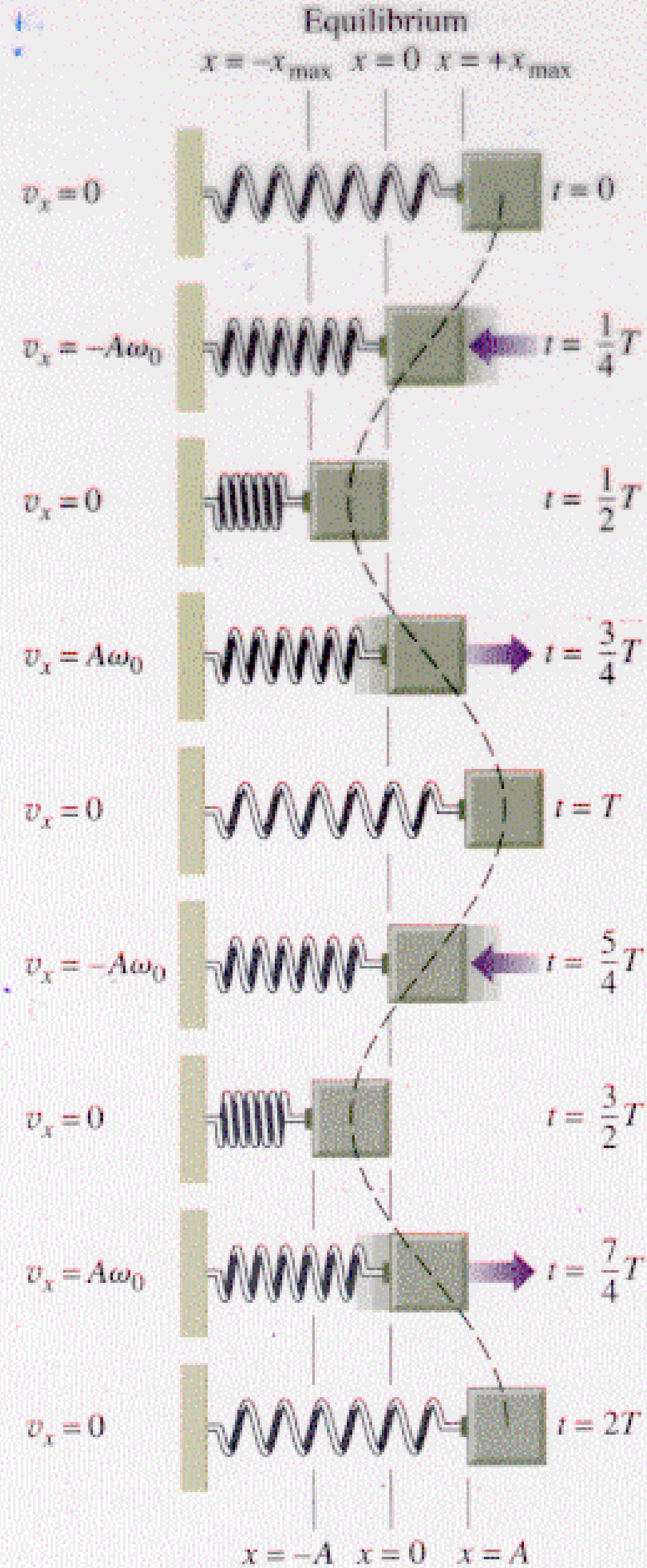
If $\phi = -\pi/2$, $x = A\cos(\omega t - \pi/2) = A\sin\omega t$

Then $x=0$ at $t=0$

$v = \frac{dx}{dt} = +A\omega$ at $t=0$ (give object a "kick") impulse

Figure 10.28

# Simple harmonic oscillator



# SHM : Speed + Acceleration

$$x = A\cos(\omega t + \phi)$$

**Speed** $v = -A\omega \sin(\omega t + \phi) = A \cdot \frac{2\pi}{T} \sin(\dots)$

- depends on amplitude and frequency

Note, using $\sin^2\theta + \cos^2\theta = 1$ to eliminate "$t$"

$$\Rightarrow v = \pm A\omega \sqrt{1 - (x/A)^2}$$

- has value $\pm A\omega$ at $x = 0$

$= 0$ at $x = \pm A$ (extremes of motion)

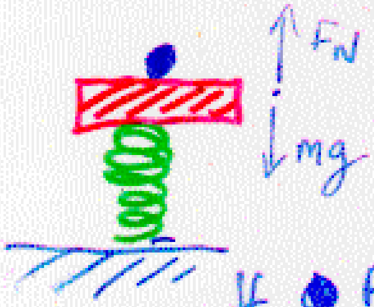
e.g. take photo of child on swing at ends of motion, not at "bottom" ($x=0$) when $v \neq 0$.

**Acceleration** $a = \frac{dv}{dt} = -\omega^2 A \cos(\dots) = -\omega^2 x$

$= 0$ at $x = 0$ (so object keeps moving)

$= \pm\omega^2 A$ at $x = \pm A$ when $v = 0$

e.g. for flea on mass on vertical spring, effective weight $F_N$



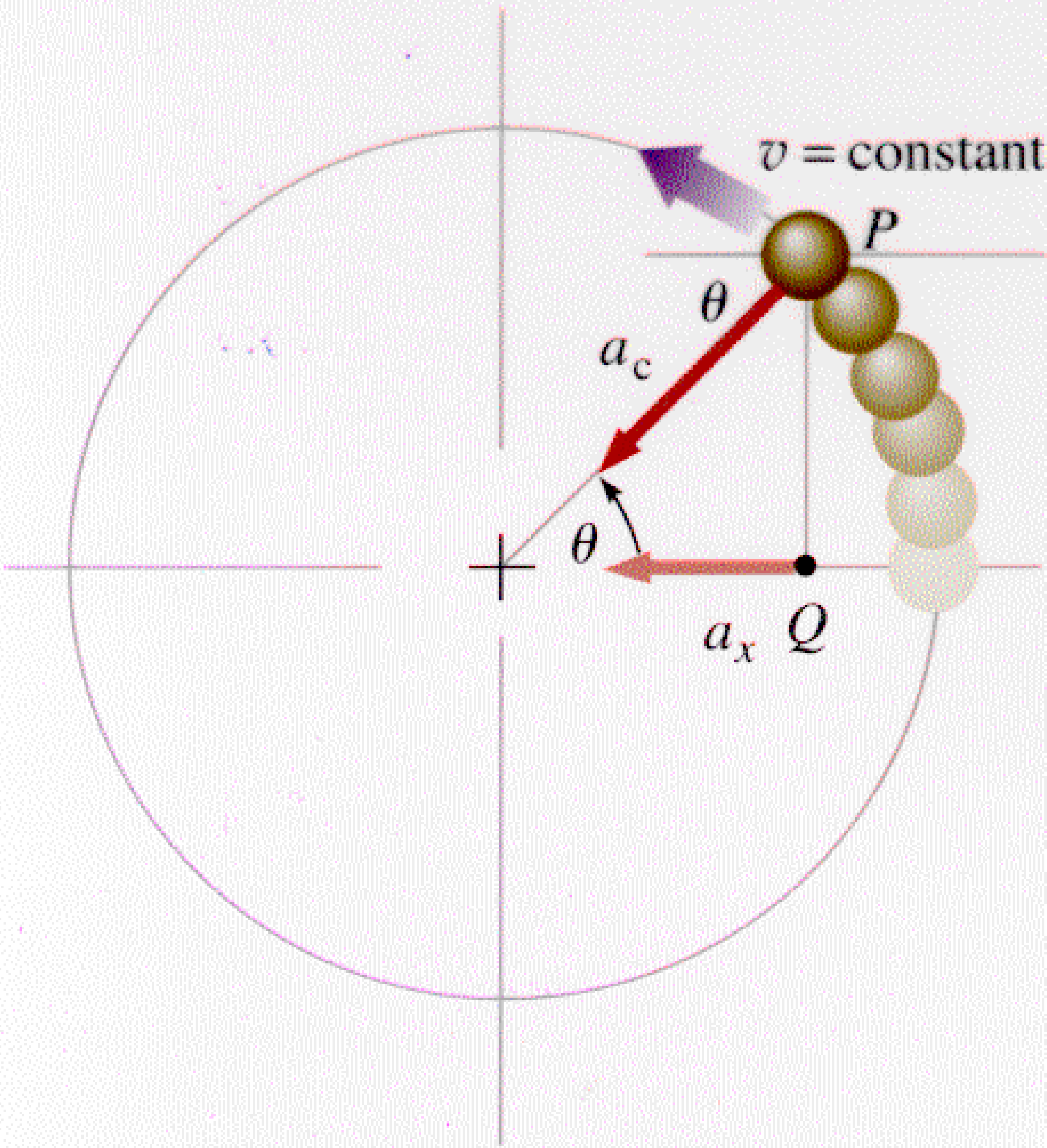

given by

$$F_N = mg \pm ma = mg \pm \omega^2 x$$

If $F_N \leq 0$, platform "leaves flea behind".

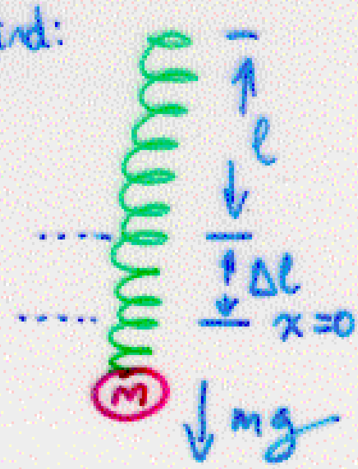
Figure 10.26

# Acceleration of an oscillator



# Example: Mass on Vertical Spring

A spring has un-extended length 0.85m. A 500g mass is attached, then let go. If $k = 10\,N/m$ ; find:




1. New eqm. length of spring
2. Amplitude + period of oscillation
3. Max. speed and accel of mass

1. New eqm. length is where $k\Delta l = mg$ i.e. $\Delta l = \frac{mg}{k} = \frac{10 \times 0.5}{10}$

$\Rightarrow$ new length $l + \Delta l = 0.85 + 0.5 = 1.35\,m$.

2. Set $x = 0$ at this length. For displacement $x$ (down)

Net force $F = m\frac{d^2x}{dt^2} = mg - k\underbrace{(\Delta l + x)}$

Now $mg = k\Delta l$ so $\Rightarrow m\frac{d^2x}{dt^2} = -kx$, oscillates about $x = 0$. (c.f. bungee jump)

Ang. freq $\omega = \sqrt{k/m} = 4.47$ rad/s. Period $T = 2\pi/\omega$

i.e. $T = 2\pi\sqrt{\frac{m}{k}} = 1.41\,s$. Note $T^2 \propto m$ (lab)

in fact we find $T^2 \propto (m + 1/3\, m_s)$

2. Since

Amplitude $A = \Delta l$ since spring returns to starting pt.

$$x = \underset{A}{0.5} \cos(\underset{\omega}{4.47} t + \phi)$$

3. Speed $v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$

Given $v = 0$ at $t = 0 \Rightarrow \phi = 0$

$v_{max} = \pm \omega A$ when $x = 0$

$= \pm 4.47 \times 0.5 = 2.23\ m/s$

4. Accel $a = -\omega^2 x$

$a_{max} = -\omega^2 A = \frac{-k}{m} A = \frac{10}{0.5\,kg} \times 0.5\,m$

$= 10\,m/s^2\ (\sim 1g)$