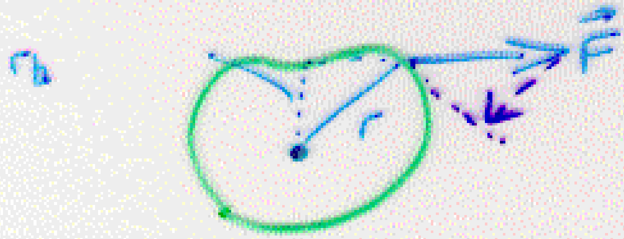
Moment of Inertia and Center of Mass

Apply a force to a rigid body at a point r from center of mass



$\Rightarrow$ object will <u>translate</u> : center of mass moves

with $a = \frac{dv}{dt} = \frac{F}{m}$

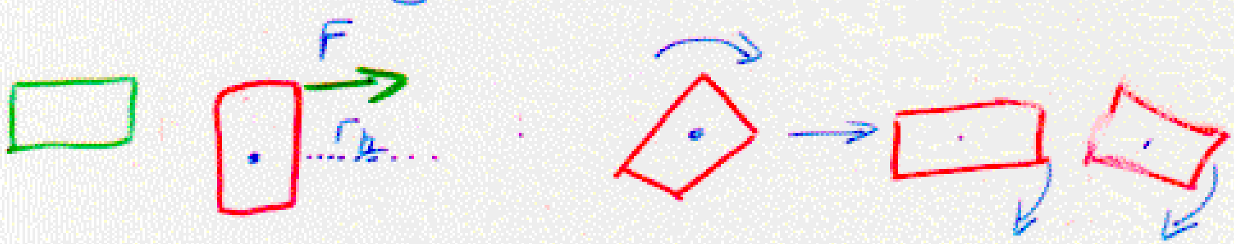
... <u>and</u> rotate about center of mass

$rF\sin\theta$

with $\alpha = \frac{d\omega}{dt} = \frac{\tau}{I} = \frac{r_\perp F}{I}$

e.g. cars crash on icy surface:

Top view:



c.f. Force applied with $r_\perp = 0$ (in line with c.m.)

Top view:

Rotational Kinetic Energy

For an object spinning at ang. speed ω about an axis

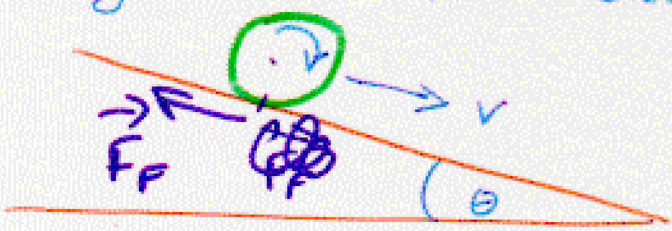
K.E. of total mass

$$KE = \sum \tfrac{1}{2} m_i v_i^2 = \int \tfrac{1}{2} v_i^2 \, dm$$

But $v_i = r_i \omega$ $\Rightarrow$ $KE = \tfrac{1}{2} \underbrace{\sum m_i r_i^2}_{= I} \omega^2$

$\Rightarrow$ Rotational KE $= \underline{\tfrac{1}{2} I \omega^2}$ (c.f. translational $= \tfrac{1}{2} m v^2$)

e.g. Roll ball, cylinder etc. down inclined plane



friction F_F provides torque
$\Rightarrow$ ball rolls

P.E lost $= mgh = \tfrac{1}{2} m v^2 + \tfrac{1}{2} I \omega^2$

If object rolls without slipping, speed of c.m.

$v = \dfrac{2\pi r}{T}$ where rot. period $T = \dfrac{2\pi}{\omega}$ $\Rightarrow$ $\underline{v = r\omega}$

$\therefore \quad mgh = \tfrac{1}{2} m v^2 + \tfrac{1}{2} \dfrac{I v^2}{r^2}$ (ex. 8.17)

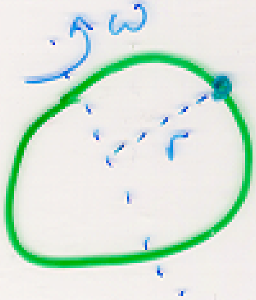
e.g. for sphere, $I = \tfrac{2}{5} m R^2$ $\Rightarrow$ $v = \sqrt{10gh/7}$

c.f. hollow sphere, $I = \tfrac{2}{3} m R^2$ $\Rightarrow$ $v = \sqrt{6gh/5}$

c.f. sliding object ($\omega = 0$) $\Rightarrow$ $\underline{v = \sqrt{2gh}}$

Translation	Rotation
Mass m	Moment of Inertia $I = \sum mr^2$ $kg\,m^2$
Force F (N)	Torque $\tau = r_{\perp} F$ (Nm)
Distance s m	Angle θ rad
Speed $v = \frac{ds}{dt}$ m/s	Angular speed $\omega = \frac{d\theta}{dt}$ rad/s
Accel. $a = \frac{dv}{dt}$ m/s^2	Angular accel. $\alpha = \frac{d\omega}{dt}$ rad/s^2
Momentum $p = mv$	Ang. momentum $L = I\omega$
K.E. $= \frac{1}{2}mv^2$	K.E. $= \frac{1}{2}I\omega^2$
$F = \frac{dp}{dt} = ma$	$\tau = \frac{dL}{dt} = I\alpha$
No net force $\Rightarrow p = \text{constant}$	No net torque $\Rightarrow L = I\omega = \text{constant}$
Equal + opposite forces	Equal + opposite torques (e.g. helicopter needs tail rotor)

Angular Momentum



For small mass m in rotating solid body :

Force $F = m\frac{dv}{dt} = mr\frac{d\omega}{dt}$ $\qquad (F = ma)$

Torque $\tau = rF = (mr^2)\frac{d\omega}{dt} = I\frac{d\omega}{dt} = I\alpha$

Define Ang. Momentum $L = I\omega$

$\Rightarrow \tau = \frac{dL}{dt} = I\alpha$ (c.f. $F = \frac{dp}{dt}$)

Note: No external torques, $\tau = 0 \Rightarrow \frac{dL}{dt} = 0$

Then ang. mom. $L = I\omega$ = constant

e.g. Spinning skater pulls in arms, reducing $I = \sum mr^2$

$\Rightarrow$ ω increases such that $L = I\omega$ = constant.

R.K.E. $= \frac{1}{2}I\omega^2 = L^2/2I$

(also, rotating stars collapse, reducing $I = \frac{2}{5}mr^2$

So $\omega = \frac{L}{I} = \frac{L}{\frac{2}{5}mr^2}$ $\uparrow$ as $r \downarrow$.