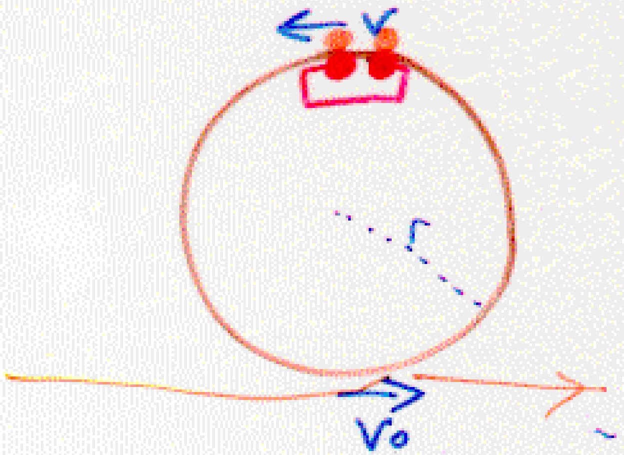
Motion in a Vertical Circle : Example

A roller-coaster enters a vertical loop of height 16m (radius $r = 8m$), unpowered. How fast must it move at the bottom in order to (a) make it to the top (b) keep passengers in their seats? (No friction).



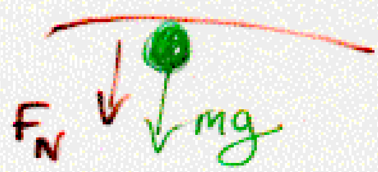
(a) to climb to height $h = 2r$

$$\overset{KE}{\tfrac{1}{2} m v_0^2} \geq \overset{PE}{mgh} = 2mgr$$

(for $v = 0$ at top)

$$\Rightarrow \quad v_0^2 \geq 4gr \quad (v > 17.9\,m/s)$$

(b) But at top of loop:



Centripetal force $\frac{mv^2}{r} = F_N + mg$ towards center

so $F_N = \frac{mv^2}{r} - mg \geq 0$ or passengers fall out!

$\Rightarrow \quad mv^2 \geq mgr$ or KE at top $\frac{1}{2} mv^2 \geq mgr/2$

$\therefore$ K.E. at top also $= \frac{1}{2} m v_0^2 - \overbrace{2mgr}^{mgh} \geq mgr/2$

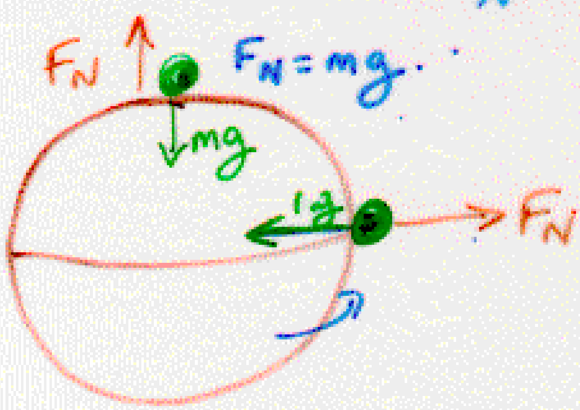
$$\Rightarrow \quad v_0^2 \geq 5gr \quad , \text{or } v_0 \geq 20\,m/s$$

c) Effective weight at bottom of loop?

Gravitation

On surface of earth, force of gravity mg = constant towards center of earth.

But earth rotates so eff. weight F_N is less at equator



$$F_c = \frac{mv^2}{r} = mg - F_N$$

$$\Rightarrow F_N = mg - \frac{mv^2}{r}$$

$\Rightarrow$ easy way to lose weight!

However, Newton and others realized that the grav. force <u>decreases</u> with distance from the earth

(compute $F_{grav} = \frac{mv^2}{r}$ for Moon's orbit)

In fact we know: $$\boxed{F_{grav} = \frac{GMm}{r^2}}$$

$\propto$ product of masses

$\propto \frac{1}{(\text{distance})^2}$

$G \equiv$ Universal Gravitational Constant

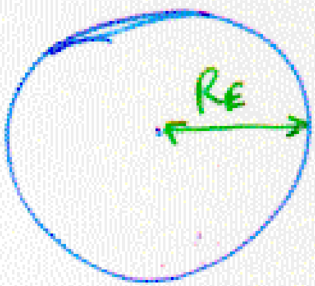
$= 6.67 \times 10^{-11}\ Nm^2/kg.$ (Cavendish) e.g. At the moon, $r = 384 \times 10^6\,m$ (Ex 5.5)

$$F_{grav} = \frac{GM_E M_M}{r^2} = 0.0027\, M_M$$

$$= 2 \times 10^{20}\,N$$

$$= \frac{M_M v^2}{r}$$ to keep moon in orbit

On surface of the earth, $r = R_E$ from center



So $F_{grav} = \dfrac{GM_E m}{R_E^2} = mg$

$\Rightarrow g = \dfrac{GM_E}{R_E^2}$ Measure g, R_E

$\Rightarrow M_E = 6 \times 10^{24}\,kg$

Similarly for surface of other planets:

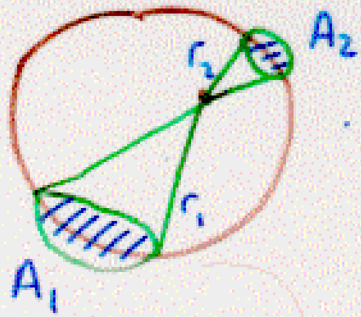
$g_{planet} = \dfrac{GM_P}{R_P^2}$ - see table 5.1

so e.g. the gas giant Saturn has $M_S >> M_E$

but $g_S = \dfrac{GM_S}{R_S^2} \approx g$ $(12\,m/s^2)$.

Gravity inside the earth - is it greater or less?

First, F_{grav} inside spherical shell:



pull from area A_2 is opposite to pull from A_1

But $A_2 \propto r_2^2$, $A_1 \propto r_1^2$

So if each area has mass $m_1 \propto A_1$, $m_2 \propto A_2$

$$F_1 = Gm \frac{m_1}{r_1^2} \propto Gm \frac{A_1}{r_1^2} \propto Gm \frac{r_1^2}{r_1^2}$$

- independent of r_1, r_2 $\Rightarrow$ all forces balance

$\therefore$ Inside spherical shell, $F_{grav} = 0$!

So as we dig into earth, at radius r from center we need only count F_{grav} from the mass within radius r. For density ρ, $M(r) = \underbrace{4/3 \pi r^3}_{Vol.} \rho$

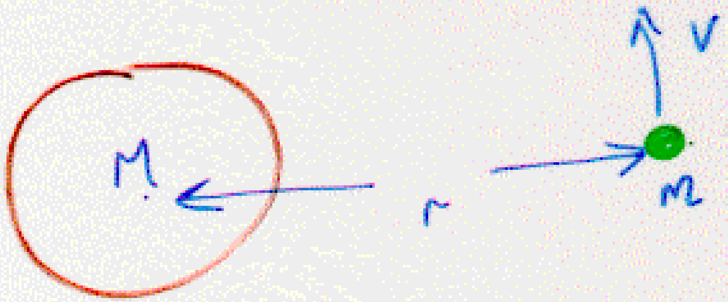
So $F_{grav} = Gm \frac{M}{r^2} = Gm \frac{4/3 \pi r^3}{r^2} = Gm \, 4/3 \pi r$

$\propto r$.

(another way to lose weight : at $r = 0$, $F_{grav} = 0$)

Planetary Motion: Circular Orbits

e.g. Sun - Earth or Earth - satellite system
($M >> m$)



$$F_{grav} = \frac{GMm}{r^2}$$

$F_{grav} = F_c$: $\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$, orbital speed.

(Measure)
Orbital Period $T = \frac{\text{circumference}}{\text{orbital speed}} = \frac{2\pi r}{v}$ so $v = \frac{2\pi r}{T}$

So $\frac{GMm}{r^2} = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 \Rightarrow \frac{GMm}{r^2} = \frac{4\pi^2 m r^2}{r T^2}$

$\Rightarrow$ Period $T^2 = \frac{4\pi^2 m r^3}{GMm}$ $\boxed{T^2 \propto r^3}$

So if earth ($r = r_E$) has $T = 1$ year,

Mercury ($r = 0.39 r_E$) has $T = (0.39)^{3/2} = 0.24$ yr

Jupiter ($r = 5.21 r_E$) $T = (5.21)^{3/2} = 11.9$ yr

(Table 5.2)